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THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

The Association of Mathematics Teachers of India (AMTI) was started in 1965 for the promotion of efforts to improve Mathematics education at all levels. A major aim of the Association is to assist practising teachers of Mathematics in schools in improving their expertise and professional skills. Another important aim is to spot out and foster Mathematical talents in the young. The Association also seeks to disseminate new trends in Mathematics education among parents and public. Other activities of the Association include consultancy services to schools in equipping the Mathematics section of their libraries, in organising children's Mathematics clubs and fairs, in setting up teacher centres in schools, in conducting Mathematics laboratory programmes, in holding practical tests in Mathematics in assisting children in participating investigational projects.

The Association holds " National Mathematical Talent Search Competition " annually and organizes Orientation Courses, Seminars and Workshops for teachers and courses for talented students. A national conference is held annually in different parts of the country for teachers to meet and deliberate on important issues of Mathematics education. Innovative teacher award has been instituted to give public recognition to enterprising and pioneering teachers of Mathematics for which entries from teachers are invited.

An award for contributions to the Mathematics Teacher relating to History of Mathematics in the context of mathematics education had been instituted by Prof.R.C.Gupta.

"The Mathematics Teacher (India)" (MT) is the official quarterly journal of the Association and is issued twice a year. It has been approved for use in schools and colleges of education by the Government departments of education in many States. Besides MT the Association also brings out Junior Mathematician (JM), three issues in a year, especially for school students in English and Tamil.

The membership of the Association is open to all those interested in Mathematics and Mathematics Education. The membership fee inclusive of subscription for "**The Mathematics Teacher (India)**" and effective from April 1993 is as follows:

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Dr. S.Muralidharan
Editor

EDITORIAL

This issue contains problems and solutions from the Mathematics competitions held during 2017.

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SCREENING TEST – GAUSS CONTEST
NMTC at PRIMARY LEVEL
V & VI Standards

PART – A

1. Which one of the following numbers is NOT the sum of two prime numbers?

   A. 24 B. 30 C. 67 D. 21

   Solution We have $24 = 17+7, 30 = 11+19, 21 = 2+19$. Since 67 is odd, if we write 67 as a sum of two prime numbers, one of them must be 2. But then, $67 - 2 = 65$ is not a prime. Thus 67 can not be written as a sum of two prime numbers and the answer is C.

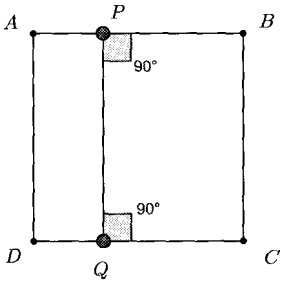

Figure 1

2. $ABCD$ is a square and $PB = 2AP$. The perimeter of the rectangle $APQD$ is 80 cm. The perimeter of $ABCD$ in cms is

   A. 100 B. 120 C. 140 D. 160

   Solution Since $PB = 2AP$, $AB = AD = 3AP$. The

perimeter of $APQD$ is

$$AP + PQ + DQ + AD = AP + 3AP + AP + 3AP$$
$$= 8AP$$

Thus $AP = 10$ and $AB = 30$. The perimeter of $ABCD$ is hence $4 \times 30 = 120$ cms. Answer is B.

3. Saket added up all the even numbers from 1 to 101. Then, from the total he obtained, he subtracted all odd numbers between 0 and 100. The answer he would have obtained is

A. 0 B. 20 C. 30 D. 50

Solution The even numbers between 1 and 101 are

$$2, 4, 6, \ldots, 100$$

and the odd numbers between 0 and 100 are

$$1, 3, 5, \ldots, 199$$

Thus

$$(2 + 4 + 6 + \cdots + 100) - (1 + 3 + \cdots + 99)$$
$$= (2 - 1) + (4 - 3) + \cdots + (100 - 99)$$
$$= 1 + 1 + \cdots + 1$$
$$= 50$$

Answer is D.

4. The value of $\frac{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}}{2 + 4 + 8}$ is

A. 16 B. 4 C. $\frac{1}{4}$ D. $\frac{1}{16}$

Solution We have

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{7}{8}$$

and hence the given fraction is $\frac{\frac{7}{8}}{14} = \frac{1}{16}$. The answer is D.

5. $ABCD$ is a rectangle. $AB = 8$ cm and $BC = 6$ cm. Q is the midpoint of AB. P, R are on AD and BC respectively such that $AP = 2$ cm, $CR = 1$ cm. Area of the shaded triangle in square cms is

   A. 12 B. 13 C. 14 D. 16

   Solution $AQ = 4$, $AP = 2$ and hence area of triangle

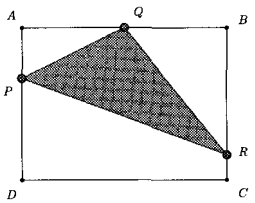

Figure 2

APQ is 4 square cms. Similarly, we find that the area of $QBR = 10$. The area of the trapezium $PRCD$ is $\frac{1}{2} \times 8 \times (4 + 1) = 20$ square cms. Since the area of the rectangle is 48 square cms, the area of the shaded part is $48 - 10 - 4 - 20 = 14$ square cms and the answer is C.

6. The *Rishimoolam* of a number is defined as follows. Consider the number 234. By multiplying its digits 2, 3 and 4, we obtain $2 \times 3 \times 4 = 24$. Again, multiplying the digits of 24, we get $2 \times 4 = 8$. We say 8 is the Rishimoolam of the number 234. If 0 is the Rishimoolam, we say the number has no Rishimoolam. Which one of the following has no Rishimoolam?

   A. 736 B. 647 C. 831 D. 619

Solution For 619, we have $6 \times 1 \times 9 = 54$, $4 \times 5 = 20$ and $2 \times 0 = 0$. Thus 619 has no Rishimoolam. For the other three nubers, we can see that the Rishimoolams are respectively $2, 6, 8$. Answer is D.

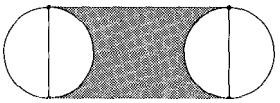

Figure 3

7. Two circles touch two parallel lines as shown in Figure 3. The radius of each circle is 1 cm. The distance between the centres of the circles is 5 cm. The area of the shaded region in square cms is

   A. 5π B. 10π C. $10 - \pi$ D. $10 + \pi$

   Solution Clearly, the distance between the centers of the squares is the longer side of the rectangle and hence equals 5 cms. The other side of the rectangle is the diameter of the circle and hence equals 2 cms. From the area of the rectangle, if we subtract the areas of the semicircles we obtain the area of the shaded region. Thus the required area is $10 - \pi$. Answer is C.

8. Samrud wrote two consecutive integers, one of which ends in a 5. He multiplied both. He squared the answer. The last two digits of his answer is

   A. 50 B. 40 C. 10 D. 00

   Solution Since one of the numbers ends with 5, the other number ends with either 4 or 6. In any case, the product of the numbers is a multiple of 10. Thus the square of

the product will be a multiple of 100 and will end with 00. Answer is D.

9. Vishwa wrote a number on each side of 3 cards. In each card, the numbers written on the sides are different. One side of each card is a prime number and the other sides had 44, 59 and 38 respectively. Given that the sum of the numbers on each card is the same, the difference between the largest and the second largest of the prime numbers on the cards is

   A. 6 B. 7 C. 9 D. 4

   Solution We need to find three prime numbers a, b, c such that $44+a, 59+b$ and $38+c$ are equal. Note that $a \neq 2$ – in fact, $a \geq 16$. Trying out the first prime after 15, we get $44 + 17 = 61$. Putting $b = 2$ and $c = 23$, we get $44 + 17 = 59 + 2 = 38 + 23 = 61$ and $17, 2, 23$ are primes. The difference between the largest and the second largest of the prime numbers is $23 - 17 = 6$ and the answer is A.

10. The number of three digit numbers abc such that $a \times b \times c = 15$ is

    A. 2 B. 6 C. 8 D. 9

    Solution The possibilities for a, b, c are $1, 3, 5$. The three digit numbers using these digits are $135, 153, 315, 351, 513, 531$ and hence we have 6 such numbers. Answer is B.

PART – B

11. Five chairs cost as much as 12 desks, 7 desks cost as much as 2 tables and 3 tables cost as much as 2 sofas. If the cost of 5 sofas is Rs 5250, then the cost of a chair (in Rs) is ———

Solution The cost of one sofa is $\frac{5250}{5} = 1050$ and hence cost of 3 tables is $2 \times 1050 = 2100$. This gives that cost of a table is 700. Cost of 7 desks is $2 \times 700 = 1400$ and hence each desk costs 200. Thus the cost of 12 desks is 2400 and this is the same as cost of 5 chairs. Thus each chair costs $\frac{2400}{5} = 480$.

12. The average age of a class of 20 children is 12.6 years. 5 new children joined with an average age of 12.2 years. The new average of the class (to one decimal place) ——
Solution The total age of the 20 children is $12.6 \times 20 = 252$. The total age of the 5 new students is $5 \times 12.2 = 61$. Hence the total age of 25 students is $252 + 61 = 313$ and the new average is $\frac{313}{25} = 12.52$ and to one decimal place the average is 12.5

13. 13 is a two digit prime and when we reverse its digits, the number 31 obtained is also a prime number. The number of two digit numbers having this property is ———
Solution The two digit prime numbers are

$$11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53,$$
$$59, 61, 67, 71, 73, 79, 83, 89, 97$$

Of these, only $11, 13, 17, 31, 37, 71, 73, 79, 97$ satisfy the condition that when we reverse the digits, we get a prime number. Thus the number of such numbers is 9.

14. In a garden there are two plants. One plant is 44 cm tall and the other is 80 cm tall. The first plant grows 3 cm in every 2 months and the second 5 cm in every 6 months. The number of months after which the two plants will have equal height is ———
Solution We tabulate the heights of the plants after 6th,

12th, ... months as follows:

Months	6	12	18	24	30	36	42	48	54
Plant 1	53	62	71	80	89	98	107	116	125
Plant 2	85	90	95	100	105	110	115	120	125

After 54 months, they reach the same height.

15. In 5 days a man walked a total of 85 KM. Every day he walked 4 KM less than the previous day. The number of KM he walked on the last day is ———

Solution Let the distances walked on the days be $d+8, d+4, d, d-4, d-8$. The sum of these distances must be 85 and hence $5d = 85$ and $d = 17$. The distance walked on the last day is $d - 8 = 17 - 8 - 9$ KM.

16. In the adjoining Figure 4, AB is parallel to CD. The value of x is ———

Solution Let AD and BC meet at E. We have

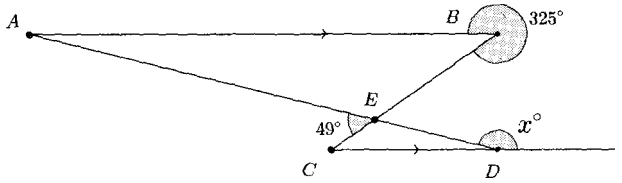

Figure 4

$\angle ADC = 180° - x$. Since AB and CD are parallel, $\angle BAE = \angle ADC = 180° - x$. Also $\angle AEB = 180° - 49° = 131°$ and $\angle ABE = 360° - 325° = 35°$. In triangle ABE, we have

$$\angle BAE + \angle ABE + \angle AEB = 180° - x + 35° + 131° = 180°$$

and hence $x = 166°$

17. In Mahadevan's cycle shop for children, there are unicycles, having only one wheel, bicycles, having two wheels and tricycles, having three wheels. Samrud counts the seats and wheels and finds that there are totally 7 seats and 13 wheels. The number of bicycles is more than tricycles. The number of unicycles in the shop is ———

Solution The number of bicycles is more than the number of tricycles. If the number of tricycles is 1, then bicycles can be $2, 3, 4, 5$ or 6. Since tricycles have 3 wheels, the number of bicycles can not be 6 (if there are 6 bicycles, the number of wheels would be $12 + 3 = 15$). Hence the number of bicycles can be $2, 3, 4, 5$. If there are 5 bicycles, then we have a total of 13 wheels but only 6 seats. Thus 5 bicycles is also impossible. Suppose that there are 4 bicycles. Then we have a total of 11 wheels among the bicycles and tricycles and if we have 2 unicycles, then number of wheels would be 13 and the number of seats would be $2+4+1 = 7$. Thus 2 unicycles is a possible solution.

Can the number of bicycles be 2 or 3? If there are 2 bicycles, then we must have 6 unicycles to make up 13 wheels but then the number of seats would be more than 7. Again, if there are 3 bicycles, then the number of unicycles should be 4 and again we have more than 7 seats. Thus the number of unicycles is 2.

18. There is a tree with several branches. Many parrots came to rest on the tree. When 6 parrots sat on each branch of the tree, all the branches were occupied but three parrots were left over. When 9 parrots sat on each branch, all parrots were seated but two branches were empty. If b is the number of branches and p is the number of parrots, the value of $b + p$ is ———

Solution Since three parrots are left when they sit 6 per branch, we must have $p = 6b + 3$. Again, when they sit 9 per branch, two branches are empty. Thus $9(b-2) = p$. Thus $9b - 18 = 6b + 3$ giving $b = 7$ and thus $p = 6 \times 7 + 3 = 45$. Hence $b + p = 52$.

19. The incomes of A and B are in the ratio $3 : 2$. Their expenditures are in the ratio $5 : 3$. If each saves Rs 10,000, then As income is (in Rs) ———

Solution Let their incomes be $3X, 2X$ and their expenditures $5Y, 3Y$ respectively. Since the amount they save is the same, we must have $3X - 5Y = 2X - 3Y$ and hence $X = 2Y$. The saving of A is $1000 = 3X - 5Y = Y$ and thus his income is $3X = 6Y = 60000$ Rs.

20. The radius of a circle is increased so that its circumference is increased by 5%. The area of the circle will increase by ———%

Solution If the radius R becomes R_1, we are given that $2\pi R_1 = 1.05 \times 2\pi R$. Thus $R_1 = 1.05R$. The area becomes

$$\pi(1.05R)^2 = \pi 1.1025R^2$$

and the percentage of change is 10.25%

SCREENING TEST – KAPREKAR CONTEST
NMTC at SUB JUNIOR LEVEL
VII & VIII STANDARDS

PART - A

1. The fraction $\frac{4}{37}$ is written in the decimal form $0.a_1a_2a_3\ldots$. The value of a_{2017} is

   A. 8 B. 0 C. 1 D. 5

   Solution Computing the first few decimal places, we have $\frac{4}{37} = 0.108108108\ldots$. Thus 108 repeats with a period of 3, and $a_{2017} = 1$ (2016 is a multiple of 3). Answer C.

2. The number of integers x satisfying the equation $(x^2 - 3x + 1)^{x+1} = 1$ is

   A. 2 B. 3 C. 4 D. 5

   Solution Consider $a^b = 1$. There are three cases: $b = 0, a \neq 0$, $a = 1$, $a = -1$ and b is an even integer. When $x+1 = 0$, we have $x^2 - 3x + 1 = 5 \neq 0$ and hence $x = -1$ is a solution.
   when $x^2 - 3x + 1 = 1$, we have two solutions $x = 0, 3$. When $x^2 - 3x + 1 = -1$, we have $(x-2)(x-1) = 0$ and thus $x = 1, 2$. When $x = 2$, $x + 1$ is odd and hence we do not have a solution. When $x = 1$, $x + 1$ is even and this is a solution. Thus we have four solutions $-1, 0, 1, 3$. Answer C.

3. The number of two digit numbers ab such that the number $ab - ba$ is a prime number is

   A. 0 B. 1 C. 2 D. 3

   Solution $ab-ba$ has value $(10a+b)-(10b+a) = 9(a-b)$ and hence is not a prime for any a, b. Answer A.

4. If $A = \frac{5425}{1444} - \frac{2987}{3045} - \frac{493}{4284}$, then

A. $1 < A < 2$ B. $2 < A < 3$ C. $3 < A < 4$
D. $A < 1$

Solution Computing the fractions to three decimal places, we get

$$\frac{5425}{1444} - \frac{2987}{3045} - \frac{493}{4284} \approx 3.757 - 0.981 - 0.115 = 2.661$$

Answer B.

5. What is the 2017^{th} letter in

$$ABRACADABRAABRACADABRA\ldots$$

where the word $ABRACADABRA$ is repeatedly written?

A. A B. B C. C D. R

Solution There are 22 letters in the word. The multiple of 22 nearest to 2017 is 2002. Hence the 2017th letter would be same as the 15th letter in the given word. Thus 2017th letter is A. Answer A.

6. How many of the following statements are true?

   (a) A 10% increase followed by another 5% increase is equivalent to a 15% increase

   (b) If the radius of a circle is doubled then the ratio of the area of the circle to the circumference is doubled

   (c) If a positive fraction is subtracted from 1 and the resulting fraction is again subtracted from 1 we get the original fraction.

A. 0 B. 1 C. 2 D. 3

Solution A 10% increase followed by another 5% increase gives $100 \times 1.1 \times 1.05 = 115.5$ and hence 15.5% increase. Thus (a) is false.

The ratio of the area of a circle of radius R to its circumference is $\dfrac{\pi R^2}{2\pi R} = \dfrac{R}{2}$. Thus when the radius is

doubled, the ratio is also doubled. Thus (b) is true. Subtracting a positive fraction f from 1 we get $1-f$ and subtracting this from 1 again, we get $1-(1-f)=f$, the original fraction. Thus (c) is true. Answer C.

7. In Figure 1, the breadth of the rectangle is 10 units. Two semicircles are drawn on the breadth as diameter. The area of the shaded region is 100 sq units. The shortest distance between the semicircles is

A. $\frac{5\pi}{2}$ B. 5π C. $\frac{5\pi}{3}$ D. $\frac{3\pi}{4}$

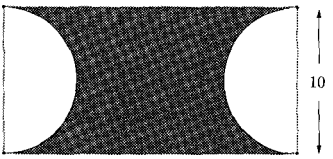

Figure 1

Solution Let the length of the rectangle be L. Area of the shaded region is $10L-25\pi$. Hence $10L-25\pi=100$ giving, $L=10+\frac{5}{2}\pi$. The shortest distance between the circles is $L-10=\frac{5}{2}\pi$. Answer A.

8. When you arrange the following in descending order

(i) 15% of 30

(ii) 8% of 15

(iii) 20% of 20

(iv) 26% of 10

(v) 9% of 25

the middle one is

A. 15% of 30

B. 8% of 15

C. 20% of 20

D. 26% of 10

Solution 15% of 30 is $0.15 \times 30 = 4.5$. Similarly, the others are $1.2, 4.0, 2.6, 2.25$. Arranging them in the descending order, we have $4.5, 4.0, 2.6, 2.25, 1.2$ and the middle one is 2.6. Answer D.

9. After simplifying the fraction

$$\left\{\frac{a+\frac{b-a}{1+ab}}{1-\frac{a(b-a)}{1+ab}}\right\}\left\{\frac{\frac{x+y}{1-xy}-y}{1+\frac{y(x+y)}{1-xy}}\right\}$$

we get a term independent of

A. a, y B. b, x C. a, b D. x, y

Solution

$$\begin{aligned}\frac{a+\frac{b-a}{1+ab}}{1-\frac{a(b-a)}{1+ab}} &= \frac{a+a^2b+b-a}{1+ab-ab+a^2}\\ &= \frac{b(1+a^2)}{1+a^2}\\ &= b\end{aligned}$$

Similarly,

$$\frac{\frac{x+y}{1-xy}-y}{1+\frac{y(x+y)}{1-xy}} = x$$

Hence the given expression evaluates to bx and is independent of a, y. Answer A.

10. A boy aims at a target shown in Figure 2. When he hits the center circle he gets 7 points, first annular region 5 points and second annular region 3 points. He shoots six times. Which one of the following is a possible score?

A. 16 B. 26 C. 19 D. 41

Solution Suppose that he shoots at the second annular region a times, first annular region b times and the

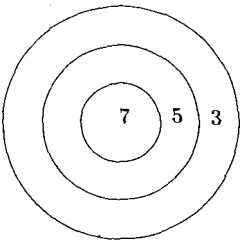

Figure 2

center circle a times. Then $a+b+c=6$ and $3a+5b+7c$ will be his total score. If $c=5$, then either $a=0, b=1$ or $a=1, b=0$. In the first case, the total score is $5\times 1+7\times 5=40$ and in the second case, the score is $3\times 1+7\times 5=38$. None of these scores is among the given scores. Let $c=4$. The possible scores are $38, 26$ or 36. Of these, 26 is a given score and hence answer is B.

11. If 7 Rasagullas are distributed to each boy of a group, 10 rasagullas would be left. If 8 are given to each boy then 5 rasagullas would be left. So the person who distributes the rasagullas brought 15 more rasagullas and distributed the same number (x) rasagullas to each. There is no rasagulla left. Then x is

    A. 10 B. 11 C. 12 D. 14

    Solution Suppose that the number of rasigullas be R and the number of boys B. We are given $7B=R-10$ and $8B=R-5$. Hence $B=5$ and $R=7B+10=45$. If 15 rasigullas are brought, the total becomes 60 and thus $x=60/5=12$. Answer C.

12. In Figure 3, all squares are of the same size. The total area of the figure is 288 square cms. The perimeter of the figure is (in cm)

A. 86 B. 96 C. 106 D. 92

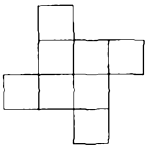

Figure 3

Solution There are eight squares in the diagram. Since the total area is 288 square cms, each individual square has area $288/8 = 36$ cms. Thus the side of the square is $x = 6$ cm. The perimeter is $16x = 96$ cms. Answer B.

13. When Newton was a primary school student he had to multiply a number by 5. But by mistake he divided the number by 5. The percentage error he committed is
A. 95% B. 96% C. 50% D. 75%
Solution If the number to be multiplied is x, then the error is $5x - x/5$ and the percentage error is

$$100 \times \frac{5x - x/5}{5x} = 96$$

Answer B.

14. ABC is an isosceles triangle with sides $AB = AC = 3x - 4 = \frac{3}{4}x + 32$. The area of the equilateral triangle with side length x is
A. $32\sqrt{3}$ B. $36\sqrt{3}$ C. $54\sqrt{3}$ D. $64\sqrt{3}$
Solution Solving for x from $3x - 4 = \frac{3}{4}x + 32$, we get $x = 16$ and hence the area of equilateral triangle with side length x is $\frac{\sqrt{3}}{4}x^2 = 64\sqrt{3}$. Answer D.

15. Two distinct numbers a and b are selected from $1, 2, 3, \ldots, 60$. The maximum value of $\frac{a \times b}{a - b}$ is

A. 6750 B. 5270 C. 4850 D. 3540

Solution We need the numerator to be as large as possible and denominator as small as possible. Clearly, $a = 60, b = 59$ yields the desired maximum as 3540. Answer D.

PART – B

16. Two cogged wheels of which one has 16 cogs and the other 27 cogs, mesh into each other. If the latter turns 80 times in three quarters of a minute, the number of turns made by the other in 8 seconds is ———

Solution When the first wheel makes 27 turns, the second will make 16 turns. Hence when the second wheel makes 80 turns, the first wheel would have $5 \times 27 = 135$ turns. Since the time taken by the second wheel to make 80 turns is 45 seconds, the first wheel makes 135 turns in 45 seconds. In 8 seconds, it would make $\frac{135}{45} \times 8 = 24$ turns.

17. If n is a positive integer such that $a^{2n} = 2$, then $2a^{6n} - 16$ is ———

Solution $a^{6n} = \left(a^{2n}\right)^3 = 8$. Thus $2a^{6n} - 16 = 0$.

18. The least number of children in a family such that every child has at least one sister and one brother is ———

Solution For any girl, we need another girl and a boy and for any boy, we need a girl and a boy. Thus the minimum number is 4.

19. A water tank is $\frac{4}{5}$ full. When 40 liters of water is removed, it becomes $\frac{3}{4}$ full. The capacity of the tank in liters is ———

Solution From the given data, if the capacity of the tank in liters is C, we have $\frac{4}{5}C - 40 = \frac{3}{4}C$ giving $C = 800$.

20. ABC is an equilateral triangle. Squares are described on the sides AB and AC as shown. The value of x is ———

Solution In the triangle ADE,

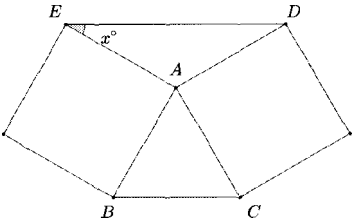

Figure 4

$$\angle EAD = 360° - 90° - 90° - 60° = 120°$$

Hence $2x = 180° - 120°$ and $x = 30°$.

21. $ABCD$ is a trapezium with $AB = 6$ cm, $AD = 8$ cm and $CD = 18$ cms. The sides AB and CD are parallel and AD is perpendicular to AB. P is the point of intersection of AC and BD. The difference between the areas of the triangles PCD and PAB in square cms is ———

Solution The area of the triangle ABD is $\frac{1}{2} \times AB \times$

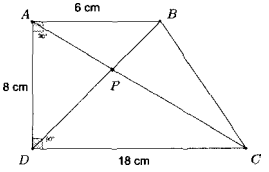

Figure 5

$AD = 24$ and the area of the triangle ACD is $\frac{1}{2} \times$

$CD \times AD = 72$. Since $\Delta ABD = \Delta PAB + \Delta APD$ and $\Delta ACD = \Delta APD + \Delta PCD$, it follows that

$$\Delta ACD - \Delta ABD = \Delta PCD - \Delta PAB = 72 - 24 = 48$$

22. The price of cooking oil has increased by 25%. The percentage of reduction that a family should effect in the use of oil so as not to increase the expenditure is ———
Solution Suppose that the current price of the oil is C per Kg and the family uses Y Kgs. The new price of the oil would be $1.25C$ and hence we have $C \times Y = 1.25 \times C \times Y_1$ where Y_1 is the new usage of the family. Hence $Y_1 = \frac{1}{1.25}Y = \frac{4}{5}Y$. Thus the usage must reduce by 20%.

23. The number of natural numbers between 99 and 999 which contains exactly one zero is ——
Solution We need to find the number of three digit numbers that have only one zero. Zero can not occur as the first digit. If the second digit is 0, then we have 9 choices for the first digit and 9 choices for the third digit. Thus we have 81 numbers that have 0 as the second digit. Similarly, we have 81 numbers with 0 as the third digit. Thus the required number of numbers is 162.

24. In Figure 6, we have semicircles and $AB = BC = CD$. The ratio of the unshaded area to the shaded area is ——
Solution Let $AB = BC = CD = 2r$. Then $AD = 6r$ and the area of the semi circle on AD as diameter is $\frac{1}{2}9\pi r^2$. Also $AC = 4r$ and the area of the semicircle on AC as diameter is $\frac{1}{2}4\pi r^2$. The area of the semi circle on AB as diameter is $\frac{1}{2}\pi r^2$. Thus the shaded area equals $\frac{1}{2}(4-1)\pi r^2 = \frac{3}{2}\pi r^2$ and the unshaded area is $\frac{1}{2}(9-3)\pi r^2 = \frac{6}{2}\pi r^2$. Thus the ratio of the unshaded to the shaded area is $2:1$.

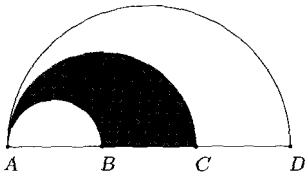

Figure 6

25. Gold is 19 times as heavy as water and copper is 9 times as heavy as water. The ratio in which these two metals be mixed so that the mixture is 15 times as heavy as water is ———

Solution Let the ratio be $x : y$. When x parts of gold and y parts of copper are mixed, then the mixture is $\frac{19x + 9y}{x + y}$ times as heavy as water. Thus

$$\frac{19x + 9y}{x + y} = 15 \Rightarrow 4x = 6y \Rightarrow \frac{x}{y} = \frac{3}{2}$$

and $x : y = 3 : 2$

26. Five angles of a heptagon (seven sided polygon) are $160°, 135°, 185°, 145°$ and $125°$. If the other two angles are both equal to $x°$, then x is ———

Solution The sum of the angles of the heptagon is $5 \times 180° = 900°$. Thus

$$2x = 900° - 160° - 135° - 185° - 145° - 125° = 150°$$

and $x = 75°$.

27. $ABCD$ is a trapezium with AB parallel to CD and AD perpendicular to AB. If $AB = 23$ cm, $CD = 35$ cm and $AD = 5$ cm. The perimeter of the given trapezium in cms is ———

Solution If E is the foot of the perpendicular from B on CD, then $BE = AD = 5$, $EC = CD - DE =$

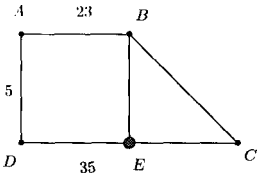

Figure 7

$35 - 23 = 12$ and hence $BC = \sqrt{5^2 + 12^2} = 13$. Thus the perimeter is $5 + 23 + 13 + 35 = 76$.

28. The number of three digit numbers which are multiples of 11 is ———

Solution The smallest three digit number that is a multiple of 11 is $110 = 10 \times 11$ and the largest three digit number that is a multiple of 11 is $990 = 90 \times 11$. Thus the number of three digit numbers that are multiples of 11 is $90 - 10 + 1 = 81$.

29. If a, b are digits, ab denotes the number $10a + b$. Similarly, when a, b, c are digits, abc denotes the number $100a + 10b + c$. If X, Y, Z are digits such that $XX + YY + ZZ = XYZ$, then $XX \times YY \times ZZ$ is ———

Solution From $XX + YY + ZZ = XYZ$, using the remainder when the sides are divided by 10, we deduce that $X + Y + Z \pmod{10} = Z$ and hence $X + Y = 0 \pmod{10}$. Note that XX, YY, ZZ are multiples of 11. Hence when we divide the left hand side by 11, we obtain the quotient $X + Y + Z$ and thus $100X + 10Y + Z = 11(X + Y + Z)$. Thus $89X = Y + 10Z$. Since the maximum value of the right hand side in this equation is 99, it follows that $X \leq 1$ and since $X \neq 0$, we have $X = 1$. Since $X + Y$ is a multiple of 10, $Y = 9$

and thus $10Z = 89X - Y = 80$ and $Z = 8$. Thus $XX \times YY \times ZZ = 11 \times 99 \times 88 = 95832$.

30. The positive integer n has 2, 5 and 6 as its factors a ıd the positive integer m has 4,8,12 as its factors. The sm ıallest value of $m + n$ is ———
Solution The smallest number with factors $2, 5, 6$ is 30 and the smallest number with factors $4, 8, 12$ is 24. Hence the smallest value of $m + n$ is 54.

SCREENING TEST – BHASKARA CONTEST
NMTC at JUNIOR LEVEL
IX & X Standards

PART - A

1. If m is a real number such that $m^2 + 1 = 3m$, the value of

$$\frac{2m^5 - 5m^4 + 2m^3 - 8m^2}{m^2 + 1}$$

is A. 1 B. 2 C. -1 D. -2

Solution

$$\begin{aligned}
2m^5 - 5m^4 + 2m^3 - 8m^2 &= m^2(2m^3 - 5m^2 + 2m - 8) \\
&= (3m-1)(2m(3m-1) - 5(3m-1) + 2m - 8) \\
&= (3m-1)(6m^2 - 2m - 15m + 5 + 2m - 8) \\
&= (3m-1)(6m^2 - 15m - 3) \\
&= (3m-1)(6(3m-1) - 15m) - 3) \\
&= (3m-1)(3m-9) \\
&= 9m^2 - 30m + 9 \\
&= 9(3m-1) - 30m + 9 = -3m
\end{aligned}$$

Hence

$$\frac{2m^5 - 5m^4 + 2m^3 - 8m^2}{m^2 + 1} = \frac{-3m}{3m} = -1$$

Answer C.

2. Consider the equation $\frac{7x}{2} - a = \frac{5x}{3} + 9$. The least positive a for which the solution x to the equation is a positive integer is

A. 1 B. 2 C. 3 D. 4

Solution

$$a = \frac{7x}{2} - \frac{5x}{3} - 9 = \frac{11x}{6} - 9$$

Thus x must be a multiple of 6 and since a is positive, it follows that the least a is $11 - 9 = 2$. Answer B.

3. If $x = 2017$ and $y = \dfrac{1}{2017}$, the value of

$$\left\{\frac{\frac{x}{y}+2}{\frac{x}{y}+1}+\frac{x}{y}\right\} \div \left\{\frac{x}{y}+2-\frac{\frac{x}{y}}{\frac{x}{y}+1}\right\}$$

is A. 2017 B. 2017^2 C. $\dfrac{1}{2017^2}$ D. 1

Solution

$$\begin{aligned}\frac{\frac{x}{y}+2}{\frac{x}{y}+1} &= \frac{x+2y}{x+y}+\frac{x}{y}\\ &= \frac{xy+2y^2+x^2+xy}{y(x+y)}\\ &= \frac{x^2+2y^2+2xy}{y(x+y)}\end{aligned}$$

and

$$\begin{aligned}\frac{x}{y}+2-\frac{\frac{x}{y}}{\frac{x}{y}+1} &= \frac{x+2y}{y}-\frac{x}{x+y}\\ &= \frac{(x+2y)(x+y)-xy}{y(x+y)}\\ &= \frac{x^2+2y^2+2xy}{y(x+y)}\end{aligned}$$

Hence the required value is 1. Answer D.

4. The ratio of an interior angle of a regular pentagon to an exterior angle of a regular decagon is
A. 4 : 1 B. 3 : 1 C. 2 : 1 D. 7 : 3

Solution The interior angle of a regular pentagon is $108°$. The exterior angle of a regular decagon is $180° - \frac{8}{10} \times 180° = 36°$. Thus the required ratio is 3 : 1. Answer B.

5. The smallest integer x which satisfies the inequality $\dfrac{x-5}{x^2+5x-14} > 0$ is

   A. -8 B. -6 C. 0 D. 1

   Solution We have

$$\frac{x-5}{x^2+5x-14} = \frac{x-5}{(x+7)(x-2)}$$

   The numerator is positive if $x > 5$ and the denominator is positive when $x > 2$ or $x < -7$. Thus both numerator and denominator are positive when the integer x is such that $x \geq 6$.
   The numerator is negative if $x < 5$ and the denominator is negative when $-7 < x < 2$. Thus both numerator and denominator are negative when the integer x is -6. Answer is B.

6. If x and y satisfy the equations

$$\sqrt{\frac{20y}{x}} = \sqrt{x+y} + \sqrt{x-y}$$
$$\sqrt{\frac{16x}{5y}} = \sqrt{x+y} - \sqrt{x-y}$$

   the value of $x^2 + y^2$ is
   A. 2 B. 16 C. 25 D. 41

   Solution Since square root of x/y is part of the equation, it follows that x and y should have the same sign. Again, square roots $x+y$ and $x-y$ also appear in the equation and we deduce that both $x+y$ and $x-y$ are positive. Thus x is positive and so is y.
   Multiplying the two equations, we get

$$\sqrt{\frac{20y}{x}}\sqrt{\frac{16x}{5y}} = \left(\sqrt{x+y} + \sqrt{x-y}\right)\left(\sqrt{x+y} - \sqrt{x-y}\right)$$

and hence $8 = (x + y) - (x - y) = 2y$ and $y = 4$. Substituting in the second equation and solving for x, we get $x = 5$. Thus $x^2 + y^2 = 41$. Answer D.

7. 125% of a number x is y. What percentage of $8y$ is $5x$?

   A. 30% B. 40% C. 50% D. 60%

   Solution Given $1.25x = y$. Thus $5x = 4y = \frac{1}{2}(8y) = 0.5 \times (8y)$. Thus $5x$ is 50% of $8y$. Answer C.

8. In the adjoining Figure 1, O is the center of the circle and $OD = DC$. If $\angle AOB = 87°$, the measure of the angle $\angle OCD$ is

   A. 27° B. 28° C. 29° D. 19°

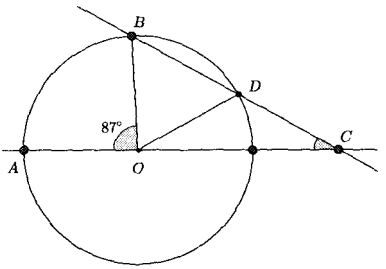

Figure 1

Solution Let $\angle OCD = x$ in degrees. Since $OD = DC$, we have $\angle DOC = \angle OCD = x$. Thus $\angle BOD = 180° - 87° - x = 93° - x$. Now, as an exterior angle of triangle ODC, $\angle BDO = 2x$. Thus in triangle BOD,

$$\begin{aligned} 180° &= \angle BOD + \angle BDO + \angle OBD \\ &= 93° - x + 2x + 2x \end{aligned}$$

Thus $3x = 87°$ and $x = 29°$. Answer C.

9. a, b, c, d, e are real numbers such that

$$\frac{a}{b} = \frac{2}{3}, \quad \frac{b}{c} = \frac{1}{3}, \quad \frac{c}{d} = \frac{1}{4}, \quad e = \frac{ac}{b^2 + c^2}$$

The value of e is

A. $\frac{1}{9}$ B. $\frac{2}{9}$ C. $\frac{1}{5}$ D. $\frac{2}{5}$

Solution We have

$$c = \frac{d}{4}, \quad b = \frac{c}{3} = \frac{d}{12}, \quad a = \frac{2b}{3} = \frac{d}{18}$$

Hence

$$e = \frac{ac}{b^2 + c^2} = \frac{(d/18) \times (d/4)}{(d/12)^2 + (d/4)^2} = \frac{1}{5}$$

Answer C.

10. The length and breadth of a rectangular field are integers. The area is numerically 9 more than the perimeter. The perimeter is

A. 24 B. 32 C. 36 D. 40

Solution Let the length and breadth be a and b respectively. Given $ab = 2(a + b) + 9$.

We have $ab - 2(a + b) = (a - 2)(b - 2) - 4$ and hence $(a-2)(b-2) = 13$. Since a, b are integers, we must have $a - 2 = 13, b - 2 = 1$. The perimeter is 36. Answer C.

11. $ABCD$ is a trapezium in which ABC is an equilateral triangle with area $9\sqrt{3}$ square units. If $\angle ADC = 90°$, the area of the trapezium in square units is

A. $12\sqrt{3}$ B. $\frac{15\sqrt{3}}{2}$ C. $\frac{27\sqrt{3}}{2}$ D. $\frac{35\sqrt{3}}{2}$

Solution The area of an equilateral triangle with side length a is $\frac{\sqrt{3}}{4}a^2$. Since the area of the triangle ABC is $9\sqrt{3}$, length of its side is 6. Also AD equals half the side of triangle ABC and hence equals 3. CD is the height of the triangle ABC and $= 3\sqrt{3}$. Hence the area of $ABCD$ is

$$9\sqrt{3} + \frac{3}{2} \times 3\sqrt{3} = \frac{27\sqrt{3}}{2}$$

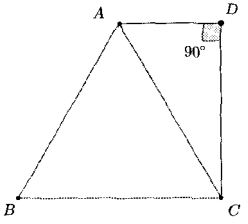

Figure 2

Answer C.

12. p is a prime number such that $p^2 - 8p - 65 > 0$. The smallest value of p is
A. 7 B. 11 C. 13 D. 17

Solution $p^2-8p-65 = (p-4)^2-81$. Thus $(p-4)^2 > 81$ and the smallest prime that satisfies this inequality is 17. Answer D.

13. The least positive integer n such that $2015^n + 2016^n + 2017^n$ is divisible by 10 is
A. 1 B. 3 C. 4 D. None of these

Solution The last digit of 2015^n is 5 for all n and the last digit of 2016^n is 6 for all n. For $n = 1, 2, 3, \ldots$, the last digits of 2017^n are $7, 9, 3, 1, 7, 9, 3, 1, \ldots$. Hence if n is such that $2015^n + 2016^n + 2017^n$ is divisible by 10, then the last digit of 2017^n must be 9. Hence $n = 2$. Answer D.

14. In a quadrant of a circle of diameter 4 units, semicircles are drawn as shown in Figure 3. The radius of the smaller circle (B) is A. 1/2 B. 1/3 C. 2/3 D. 3/4

Solution Let the centers of the quadrant, circle A and circle B be respectively, O, P and Q. Let r be the radius of the circle B. If R is the point of contact of the

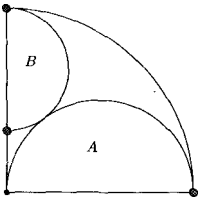

Figure 3

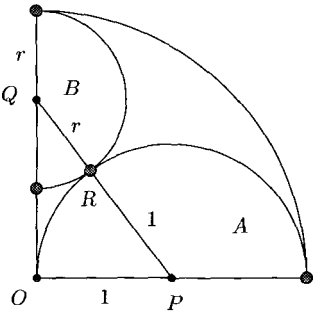

Figure 4

circles A and B, P, R, Q are collinear, $QR = r$ and $PR = 1$. Now $OQ = 2 - r$ and from the right angled triangle OPQ, we have $(1 + r)^2 = 1^2 + (2 - r)^2$ and $r = 2/3$. Answer C.

15. The product of two positive integers is twice their sum; the product is also equal to six times the difference between the two integers. The sum of these integers is
A. 6 B. 7 C. 8 D. 9

Solution Let the integers be m, n. Given $mn = 2(m + n)$ and $mn = 6(m - n)$. Hence $\dfrac{m+n}{m-n} = 3$ and $\dfrac{2m}{2n} = \dfrac{4}{2} = 2$. Thus $m = 2n$. Substituting in $mn = 2(m + n)$, we get $2n^2 = 6n$ and $n = 3$. It follows that $m = 6$ and $m + n = 9$.

Answer D.

PART – B

16. n is a natural number such that n minus 12 is the square of an integer and n plus 19 is the square of another integer. The value of n is ———

Solution Let $n - 12 = x^2$ and $n + 19 = y^2$, where x, y are integers. Then $31 = y^2 - x^2 = (y + x)(y - x)$. Since 31 is a prime, we have $y+x = 31, y-x = 1$. Thus $y = 16$ and $n = y^2 - 19 = 256 - 19 = 237$.

17. The number of there digit numbers which have odd number of factors is ———

Solution Given an integer n, we can factorize n as $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $p_1, p_2, \ldots, p_k$ are primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers. The number of factors of n is given by $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$. If n has odd number of factors, then all α_i must be even. Thus n is a product of even powers of primes and thus is a perfect square. The least three digit square is 100 and the largest three digit square is $31^2 = 961$. Thus all the three digit squares are

$$10^2, 11^2, 12^2, \ldots, 31^2$$

and there are 22 of these.

18. The positive integers a, b, c are connected by the inequality

$$a^2 + b^2 + c^2 + 3 < ab + 3b + 2c$$

then the value of $a + b + c$ is ———

Solution We have

$$2a^2 + 2b^2 + 2c^2 + 6 < 2ab + 6b + 4c$$
$$(a - b)^2 + a^2 + (b - 3)^2 + 2(c - 1)^2 < 5$$
$$(a - b)^2 + a^2 + (b - 3)^2 + (c - 1)^2 + (c - 1)^2 < 5$$

Since a, b, c are natural numbers, each of the term can have values $0, 1, 4$. Also not all can be non zero. If $c - 1 = 0$, then $a^2 + (a - b)^2 + (b - 3)^2 < 5$. If $a = 0$, this gives $b^2 + (b - 3)^2 < 5$ and $(b - 1)(b - 2) < 0$ and $1 < b < 2$. This is not possible since b is a natural number. Thus $a \neq 0$. If $a = 1$, we get $(b-1)(b-3) < 0$ and hence $b = 2$. In this case, $a + b + c = 4$. When $a = 2$, we get $(b - 2)(b - 3) < 0$, and this is impossible. If $(c - 1)^2 = 1$, then we easily see that there are no possible values for a, b. Thus the value of $a+b+c = 4$.

19. The sum of all roots of the equation $|3x - |1 - 2x|| = 2$ is ———

Solution

$$|3x - |1 - 2x|| = \begin{cases} x + 1, & \text{if } x \geq 1/2; \\ 5x - 1, & \text{if } 1/5 \leq x \leq 1/2; \\ 1 - 5x, & \text{if } x \leq 1/5 \end{cases}$$

The graph of the function $f(x) = |3x - |1 - 2x|| - 2$ is shown in Figure 5. Thus the roots are $x = 1$ and

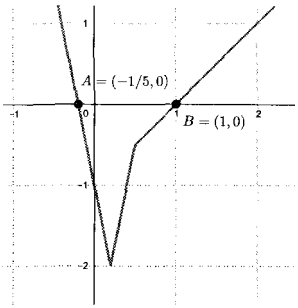

Figure 5

$x = -1/5$. The sum of the roots is $4/5$.

20. PQR is a triangle with $PQ = 15, QR = 25, RP = 30$. A, B are points on PQ and PR respectively such that $\angle PBA = \angle PQR$. The perimeter of the triangle PAB is 28, then the length of AB is ———

Solution Triangles PAB and PQR are similar. Thus

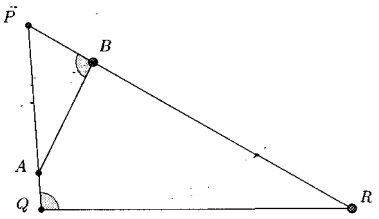

Figure 6

$$\frac{AB}{QR} = \frac{PB}{PQ} = \frac{PA}{PR} = \frac{AB + PB + PA}{QR + PQ + PR} = \frac{28}{70}$$

and $AB = QR \times \frac{2}{5} = 10$.

21. A hare sees a hound 100 m away from her and runs off in the opposite direction at a speed of 12 KM an hour. A minute later the hound perceives her and gives a chase at a speed of 16 KM an hour. The distance at which the hound catches the hare (in meters) is ———

Solution After one minute, the distance between the hound and the hare is $100 + \frac{12000}{60} = 300$ meters. The relative speed is 4 KM per hour and the hound catches the hare after $300 \times \frac{60}{4000}$ minutes. Thus the distance at which the hound catches the hare is

$$300 \times \frac{60}{4000} \times \frac{16000}{60} = 1200$$

meters.

22. Two circles touch both the arms of an angle whose measure is $60°$. Both the circles also touch each other externally. The radius of the smaller circle is r. The radius of the bigger circle (in term of r) is ———
Solution Let A, B be the centers of the circles and let

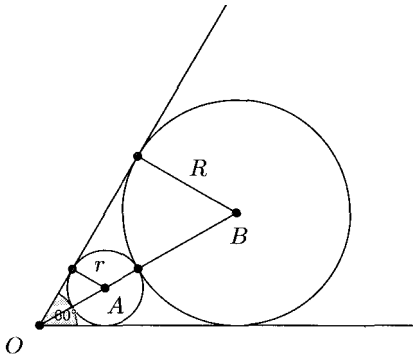

Figure 7

R be the radius of the bigger circle. The line OAB bisects the angle and hence $OA = 2r$, $OB = 2R$. Since the circles touch each other externally, $AB = R + r = OB - OA = 2R - 2r$ and $R = 3r$.

23. a, b are distinct natural numbers such that $\frac{1}{a} + \frac{1}{b} = \frac{2}{5}$. If $\sqrt{a+b} = k\sqrt{2}$, the value of k is ———
Solution $5(a+b) = 2ab$. Thus 5 divides ab. Suppose that $5 \nmid a$. Then $a = 5c$ where c is an integer. Substituting in the first equation, we obtain $5c + b = 2bc$ and hence $5c = b(2c-1)$. If $c = 1$, then $b = 5$ and $a = 5$ but a, b are distinct. Thus $c > 1$. Since c can not divide $2c - 1$, it divides b. Let $b = cd$. We now have $5 = d(2c-1)$. If $d = 5$, we get $c = 1$. Thus $d = 1, c = 3, a = 15, b = 3$. Hence $\sqrt{a+b} = \sqrt{18} = 3\sqrt{2}$ and $k = 3$.

24. The side AB of an equilateral triangle ABC is produced to D such that $BD = 2AC$. The value of $\dfrac{CD^2}{AB^2}$ is ——

Solution Since $\angle CBD = 120°$, by Cosine formula,

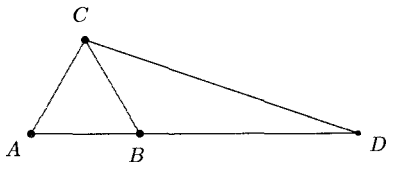

Figure 8

$$CD^2 = CB^2 + BD^2 - 2 \times CB \times BD \cos 120°$$
$$= AB^2 + 4AB^2 + 4AB^2 = 7AB^2$$

Thus $\dfrac{CD^2}{AB^2} = 7$.

25. D and E trisect the side BC of a triangle ABC. DF is drawn parallel to AB meeting AC at F. EG is drawn parallel to AC meeting AB at G. DF and EG cut at H. Then the numerical value of $\dfrac{\text{Area } (ABC)}{\text{Area } (DHE) + \text{Area } (AFHG)}$ is ——

Solution Clearly, G is a point of trisection of AB and F is a point of trisection of AC. (See Figure 9). The line GF is parallel and equal to DE. Thus triangles DEH, GHF and AGF all have equal areas. Also, it is easy to see that area of triangle DEH is $\frac{1}{9}$ of the area of triangle ABC. Thus the value of the required ratio is 3.

26. In an examination 70% of the candidates passed in English, 65% passed in Mathematics, 27% failed in both

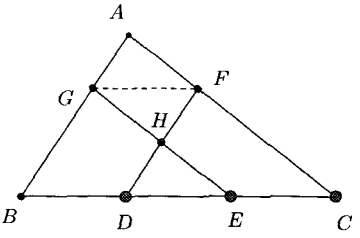

Figure 9

the subjects and 248 passed in both the subjects. The total number of candidates is ———

Solution Let E denote the set of candidates who passed in English, M, the set of candidates who passed in Mathematics. Let N be the total number of candidates. Given

$$\#(E) = 0.7N, \qquad \#(M) = 0.65N,$$
$$\#(E^c \cap M^c) = 0.27N, \qquad \#(E \cap N) = 248$$

Now,

$$\#(E \cup M) = N - \#(E^c \cap M^c) = 0.73N$$

Also,

$$\begin{aligned}\#(E \cup M) &= \#(E) + \#(M) - \#(E \cap M) \\ &= 0.7N + 0.65N - 248\end{aligned}$$

Thus $0.73N = 0.7N + 0.65N - 248$ and hence $0.62N = 248$ and $N = 400$.

27. In a potato race, a bucket is placed at the starting point, which is 7 m from the first potato. The other potatoes are placed 4 m a part in a straight line from the first one.

There are n potatoes in the line. Each competitor starts from the bucket, picks up the nearest potato, runs back with it, drops in the bucket, runs back to pick up the next potato, runs to the bucket and drops it and this process continues till all the potatoes are picked up and dropped in the bucket. Each competitor ran a total of 150 m. The number of potatoes is ———

Solution The distances of the potatoes are

$$7, 7+4, 7+2\times 4, \ldots, 7+4(i-1), \ldots, 7+4(n-1)$$

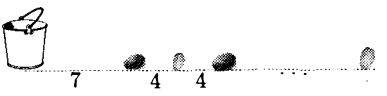

Figure 10

Hence the total distance walked to gather the n potatoes is

$$\begin{aligned} 2(7+11+15+\cdots+(7+4(n-1))) \\ &= n(14+4(n-1)) \\ &= 10n+4n^2 \end{aligned}$$

Given that $10n+4n^2 = 150$ and hence $2n^2+5n-75 = 0$. Since $2n^2+5n-75 = (2n+15)(n-5)$, it follows that $n = 5$.

28. A two digit number is obtained by either multiplying the sum of its digits by 8 and adding 1, or by multiplying the difference of its digits by 13 and adding 2. The number is ———

Solution Let the two digit number be ab. Then $10a+$

$b = 8(a + b) + 1 = 13|a - b| + 2$. Hence $2a - 7b = 1$. It follows that $a > b$ and $10a + b = 13(a - b) + 2$. This gives $3a - 14b = -2$. Solving, we get $a = 4, b = 1$ and the number is 41.

29. The inradius of a right angled triangle whose legs have lengths 3 and 4 is ———

Solution The length of the hypotenuse is 5 and the semi perimeter, $s = 6$. If r is the inradius iss r, then rs equals the area of the triangle and hence $rs = 6$ and $r = 1$.

30. a, b are such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$. If $\left(\frac{a}{b}\right)^3 + \left(\frac{b}{a}\right)^3 = 2\sqrt{n}$, where n is a natural number, the value of n is ———

Solution From $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$, we get $a^2 + ab + b^2 = 0$ and hence both a/b and b/a satisfy $t^2 + t + 1 = 0$. Thus $t^3 - 1 = (t - 1)(t^2 + t + 1) = 0$ and it follows that $(a/b)^3 = (b/a)^3 = 1$. Hence $(a/b)^3 + (b/a)^3 = 2$ and $n = 1$.

SCREENING TEST – RAMANUJAN CONTEST
NMTC at INTER LEVEL
XI & XII Standards

PART – A

1. The equation $5x^2 - 10x\cos\alpha + 7\cos\alpha + 6 = 0$ has two identical roots. If α is one of the angles of a parallelogram with sum of the lengths of two adjacent sides equal to 6, the maximum area of the parallelogram is

   A. 32/5 B. 34/5 C. 36/5 D. 18/5

   Solution: The discriminant of the given quadratic is zero. Thus

   $$100\cos^2\alpha = 20(7\cos\alpha + 6)$$

   This gives

   $$5\cos^2\alpha - 7\cos\alpha - 6 = 0$$

   and $\cos\alpha = -\frac{3}{5}$ (the other root is 2 and $\cos\alpha = 2$ is impossible). If the lengths of the two adjacent sides of the parallelogram are a, b, then $a + b = 6$ and area is $ab\sin\alpha$. This is maximum when $a = b = 3$ and the maximum area is $9\sin\alpha = \frac{36}{5}$. Answer C.

2. If $f(x)$ is a polynomial of degree three with leading coefficient 1 such that $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, then the value of $f(4)$ is

   A. 16 B. 22 C. 36 D. None of these

   Solution: Consider the polynomial $g(x) = f(x) - x^2$. Clearly, $g(x)$ is also a polynomial of degree 3 with leading coefficient 1 and $g(1) = g(2) = g(3) = 0$. Thus

$g(x) = (x-1)(x-2)(x-3)$ and $g(4) = 6$. Since $f(x) = g(x) + x^2$, it follows that $f(4) = 6 + 16 = 22$. Answer B.

3. The three roots of the equation $3x^3 + px^2 + qx - 4 = 0$ are the side length, inradius and the circumradius of an equilateral triangle. Then the value of $2p + q$ is
A. -10 B. 10 C. -12 D. 12

Solution: If a is the length of the side of the equilateral triangle, then its inradius is $\frac{a}{2\sqrt{3}}$ and its circum radius is $\frac{a}{\sqrt{3}}$. We have, by Vieta's formulas,

$$a + \frac{a}{2\sqrt{3}} + \frac{a}{\sqrt{3}} = -\frac{p}{3}$$
$$a^2\left(\frac{1}{2\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{6}\right) = \frac{q}{3}$$
$$\frac{1}{6}a^3 = \frac{4}{3}$$

Hence

$$a = 2, p = -6\left(1 + \frac{3}{2\sqrt{3}}\right), q = 12\left(\frac{1}{6} + \frac{3}{2\sqrt{3}}\right)$$

and $2p + q = -10$. Answer A.

4. Given

$$a = -\sqrt{99} + \sqrt{999} + \sqrt{9999}$$
$$b = \sqrt{99} - \sqrt{999} + \sqrt{9999}$$
$$c = \sqrt{99} + \sqrt{999} - \sqrt{9999}$$

the value of

$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)}$$

A. 22194 B. $\sqrt{99}+\sqrt{999}+\sqrt{9999}$ C. 22190
D. $\sqrt{99\times 999\times 9999}$

Solution: Since the given expression is symmetric in a,b,c, the result must be a symmetric second degree expression in a,b,c. Thus we can write

$$\frac{a^4}{(a-b)(a-c)}+\frac{b^4}{(b-c)(b-a)}+\frac{c^4}{(c-a)(c-b)}$$
$$=\lambda(a^2+b^2+c^2)+\mu(ab+bc+ca)$$

where λ,μ are constants. Putting $a=0,b=1,c=2$, we get $5\lambda+2\mu=7$ and putting $a=0,b=1,c=-1$ we get $2\lambda-\mu=1$. Thus $\lambda=\mu=1$ and the given expression equals $a^2+b^2+c^2+ab+bc+ca$.
Now,

$$\begin{aligned}a^2+b^2+c^2+ab+bc+ca&=a(a+b)+b(b+c)+c(c+a)\\&=2\sqrt{9999}a+2\sqrt{99}b+2\sqrt{999}c\\&=2(99+999+9999)\\&=22194\end{aligned}$$

Answer A.

5. In Figure 1, ABC is a right angled triangle with $\angle B=90^\circ$, $AB=8$ cm and $BC=6$ cm. Squares $ANMC, AEDB, BQPC$ are described as shown, on the sides AC, AB, BC respectively. Among PM, NE and DQ,

   A. Two are of integral length and the length of the other is irrational

   B. Two are of irrational length and the length of the other is integral

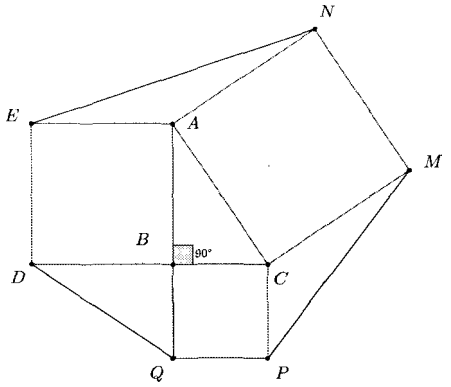

Figure 1

C. All have irrational lengths

D. All have integral length

Solution: By Pythagoras theorem, $AC = \sqrt{6^2 + 8^2} = 10$. Also,

$$\angle EAN = 360° - 90° - 90° - \angle BAC = 180° - \angle BAC$$

Hence

$$\begin{aligned} EN^2 &= AE^2 + AN^2 - 2AE \times AN \times \cos\angle EAN \\ &= 8^2 + 10^2 + 2 \times 8 \times 10 \times \frac{AB}{AC} \\ &= 64 + 100 + 128 = 292 \end{aligned}$$

Similarly,

$$\begin{aligned} PM^2 &= 6^2 + 10^2 - 2 \times 6 \times 10 \cos\angle PCM \\ &= 36 + 100 + 72 = 208 \end{aligned}$$

Clearly, $DQ = \sqrt{BC^2 + BQ^2} = \sqrt{6^2 + 8^2} = 10$. Thus DQ has integral length and the other two have irrational lengths. Answer B.

6. In Figure 2, $ABCD$ and $PQRS$ are squares. If the length of the side of the bigger square is a, the length of the side of the smaller square is

A. $\frac{a}{3}$ B. $\frac{a}{4}$ C. $\frac{a}{5}$ D. $\frac{a}{6}$

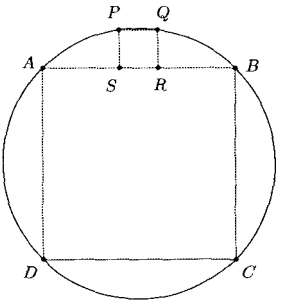

Figure 2

Solution: Let O be the center of the circle and r, its

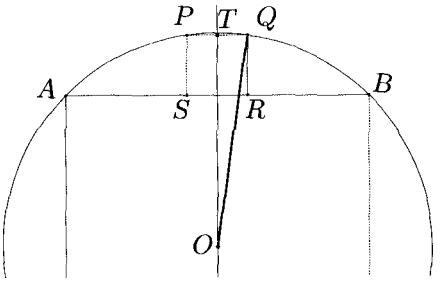

Figure 3

radius. Then $r\sqrt{2} = a$. Let $2x$ be the side of the smaller square. Let T be the foot of the perpendicular from O on PQ. From the right angled triangle OQT, we have

$OQ^2 = OT^2 + QT^2$. Thus

$$r^2 = \left(\frac{a}{2} + 2x\right)^2 + x^2$$
$$\frac{a^2}{2} = \frac{a^2}{4} + 2ax + 5x^2$$

Hence $20x^2 + 8ax - a^2 = 0$ and $(2x + a)(10x - a) = 0$. Thus $2x = \frac{a}{5}$. Answer C.

7. Twelve people sit around a circular table. Each observes that his age (viewed as an integer) is the average of the ages of his left and right neighbours. Which of the following could be the sum of their ages?
A. 224 B. 226 C. 227 D. 228

Solution: Suppose the people are $A_0, A_1, \ldots, A_{11}$. We claim that they must all have the same age. If some one, say A_0 is sitting next to two people of different ages, then either A_1 or A_{11} must be older than A_0. Suppose A_1 is older. This gives A_2 must be older than A_1, since his age is the average of the ages of A_0 and A_2 and A_0 is younger than A_1. Similar argument implies that A_3 is older than A_2, and so on, and ultimately, A_{11} is older than A_0. But that both A_1 and A_{11} are older than A_0 and the average of their ages could not be equal to that of A_0. Thus all their ages are equal and the sum of their ages must be a multiple of 12. Only 228 is a multiple of 12 among the given numbers. Answer D.

8. A rectangular billiard table has vertices at $(0,0), (12,0)$, $(0,10), (12,10)$. There are pockets only in the four corners. A ball is hit from the corner $(0,0)$ along the line $y = x$ and bounces off several walls before eventually entering a pocket. The number of walls that the ball

bounces off before entering a pocket is

A. 7 B. 8 C. 9 D. 11

Solution: Consider the tiling of the plane with the

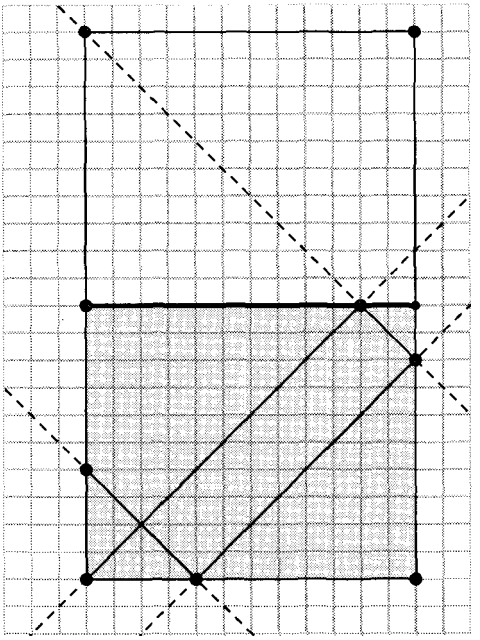

Figure 4

12×10 rectangle to form a grid. Then the reflection of the ball off a wall is equivalent to traveling along the straight line $y = x$ into another 12×10 rectangle. Hence we want to find the number of walls of the grid that the line $y = x$ hits before it reaches the first corner it hits, $(60, 60)$.

The line $y = x$ hits each of the horizontal lines $y = 10, 20, 30, 40, 50$ and each of the vertical lines $x = 12, 24, 36, 48$. This gives a total of 9 walls hit before entering a pocket. Answer C.

9. In a quadrilateral $ABCD$, we have $AB = 8, BC = 5, CD = 17$ and $DA = 10$. The diagonals AC and BD meet at E. If $BE : ED = 1 : 2$, the area of the quadrilateral $ABCD$ is

   A. 70 B. 60 C. 50 D. Not uniquely determined

   Solution: Since $BE : ED = 1 : 2$, $\Delta ABC : \Delta ACD =$

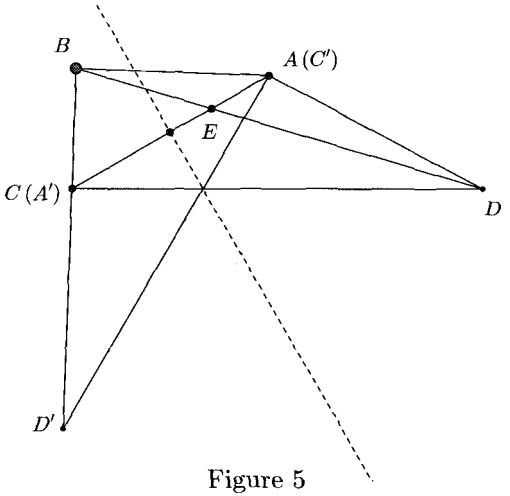

Figure 5

$1 : 2$. Reflect the triangle ACD on the perpendicular bisector of AC. In the reflected triangle $A'C'D'$, we have $AD = A'D' = 10$. The triangles ABC and $C'BD'$ have a common vertex $A(C')$ and bases BC and $A'D'$. Since $BC = 10$, their areas and bases are in the ratio $1 : 2$. Hence B, C, D' are collinear. Now, $AD' = CD = 17$ and the triangle ABD' has side lengths $8, 15, 17$. Thus $\angle ABD' = 90°$ and the area of the quadrilateral $ABCD = \frac{1}{2} AB \times BD' = 60$. Answer B.

10. Let $S(n)$ denote the sum of the digits of the integer n when n is written in the usual decimal form. For example, $S(123) = 6$. If $S(n) = 1274$, which of the

given values is *NOT* not a possible value of $S(n+1)$?
A. 1239 B. 1266 C. 1275 D. 1284

Solution: When we add 1 to a number, sum of digits either increases by 1 or reduces by $1+9k$ when the number ends with 9. For example, $S(123)=6, S(124)=7$ but $S(129)=12$ and $S(130)=4$. Of the choices given 1284 is impossible. All others are possible:

$$n=\underbrace{111\cdots1}_{1238\,ones}10099999, \qquad S(n+1)=1239$$

$$n=\underbrace{111\cdots1}_{1265\,ones}9, \qquad S(n+1)=1266$$

$$n=\underbrace{111\cdots1}_{1274\,ones}, \qquad S(n+1)=1275$$

Answer D.

11. The polynomial $g(x)=x^3+ax^2+x+10$ has three distinct roots and each root of $g(x)=0$ is also a root of $f(x)=x^4+x^3+bx^2+100x+c$. What is $f(1)$?
A. -5005 B. -6006 C. -7007 D. -8008

Solution: Let the other root of $f(x)=0$ be $-d$. Then we have $g(x)(x+d)=f(x)$

$$(x^3+ax^2+x+10)(x+d)=x^4+x^3+bx^2+100x+c$$

Equating corresponding coefficients,

$$a+d=1,\ 1+ad=b,\ d+10=100,\ 10d=c$$

Thus $d=90$, $c=900$, $a=-9$, $b=-8009$ and

$$f(1)=1+1-8009+100+900=-7007$$

Answer C.

12. The number

$$N = 1234567891011\ldots41424344$$

is the 79 digit number obtained by writing the numbers from 1 to 44 in order. What is the remainder when N is divided by 45?

A. 1 B. 9 C. 18 D. 27

Solution: Clearly, $N = 4 \pmod 5$ and since the sum of digits of N is $\frac{44 \times 45}{2} = 22 \times 45 = 0 \pmod 9$, it follows that $N = 0 \mod 9$. It suffices to find the number r such that $0 \leq r < 45$ such that $r = 4 \pmod 5 = 0 \pmod 9$. The numbers $4 \pmod 5$ are

$$4, 9, 14, 19, 24, 29, 34, 39, 44$$

and of these only 9 is a multiple of 9. Thus $N = 9 \pmod{45}$. Answer B.

13. Define the sequence F_n recursively as follows: $F_0 = 0, F_1 = 1$ and for $n \geq 2$, F_n is the remainder of $F_{n-1} + F_{n-2}$ when divided by 3. What is the value of $\sum_{k=2017}^{2024} F_k$?

A. 6 B. 7 C. 8 D. 9

Solution: The sequence is

$$0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, \ldots$$

Thus we get $F_n = F_{n+8}$. Thus

$$\sum_{k=2017}^{2024} F_k = \sum_{k=1}^{8} F_k = 9$$

Answer D.

14. P, Q, R, S have integer coordinates and are distinct points on the circle $x^2+y^2 = 25$. The distances PQ and RS are irrational numbers. What is the largest possible value of PQ/RS?
A. 3 B. $5\sqrt{2}$ C. 7 D. $3\sqrt{5}$

Solution: Taking $P = (-3,4)$, $Q = (4,-3)$, $R = (3,4)$, $S = (4,3)$, we see that the 7 is the largest possible value. Answer C.

15. In how many ways can 1000 be written as a sum of 2s and 3s, ignoring order ($500 \times 2 + 0 \times 3$ and $50 \times 2 + 300 \times 3$ are two of the ways)?
A. 500 B. 499 C. 167 D. 166

Solution: We want the number of solutions (x, y) in non negative integers to $1000 = 2x + 3y$. Noting that y must be even, we have $y = 0, 2, 4, \ldots, 332$. The number of solutions is $332/2+1 = 167$ (the $+1$ is to account for 0). Answer C.

PART – B

16. The six digit number $789ABC$ consists of six *distinct* digits and is divisible by $7, 8$ and 9. The three digit number ABC is ———

Solution 1 Since the digits are distinct, $789ABC \leq 789654$. The least common multiple of $7, 8, 9$ is 504 and hence $789ABC$ must be a multiple of 504. $\left\lfloor \frac{789654}{504} \right\rfloor = 1566$ and $1566 \times 504 = 789264$. Clearly, 789264 satisfies the conditions of the problem and hence $ABC = 264$. Since $1565 \times 504 = 788760$, no other number will satisfy the conditions of the problem.

Solution 2 We know that $A, B, C < 7$, the three digit

number ABC must be a multiple of 8 and $7+8+9+A+B+C$ must be a multiple of 9. Thus $A+B+C+6$ must be a multiple of 9. Since $A+B+C \leq 15$, it follows that $A+B+C$ is 3 or 12. Since A, B, C are distinct, $A+B+C \neq 3$. Thus $A+B+C = 12$. The possibilities of ABC are $516, 156, 615$ or any permutation of 642. Since $516, 156, 615$ are not multiples of 8, these are impossible values for ABC. Again, among the permutations of 642, the multiples of 8 are $264, 624$. Of these, 789624 is not a multiple of 7. Hence the only possibility is 789264 and $ABC = 264$.

17. m, n are relatively prime positive integers such that $\frac{m}{n} = \frac{2(\sqrt{2}+\sqrt{10})}{5\sqrt{3+\sqrt{5}}}$, then $m+n$ equals ———
Solution Squaring both sides, we get

$$\frac{m^2}{n^2} = \frac{4(12+4\sqrt{5})}{25(3+\sqrt{5})} = \frac{16}{25}$$

Since m, n are relatively prime, $m = 4, n = 5$ and $m+n = 9$.

18. ABC is a triangle with $AB = 17$ units. F is the mid point of AB and $CF = 8$ units. The maximum possible area of the triangle ABC is ———
Solution Let h be the length of the altitude from C on AB. Area of $ABC = \leq \frac{1}{2}h \times AB \leq \frac{1}{2}CF \times AB = 68$. Equality occurs when $CF \perp AB$. Thus the maximum possible area is 68.

19. Given that $a+b+c = 5$ and $1 \leq a, b, c \leq 2$ the minimum possible value of $\frac{1}{a+b} + \frac{1}{b+c}$ is ———
Solution If $a > 1$ and $b < 2$, we can decrease the sum by decreasing a and increasing b. We can follow a

similar procedure if $c > 1$ and $b < 2$. Therefore, the sum is minimized when $b = 2$. We can then cross-multiply the two fractions and see that we are trying to minimize $\frac{a+c+4}{(a+2)(c+2)}$. The product of two numbers with a fixed sum is maximized when those two numbers are equal, so $\frac{7}{(a+2)(c+2)}$ is minimized for $a = c = \frac{3}{2}$, which gives us an answer of $\frac{4}{7}$.

20. The number of ways we can insert $+$'s between the digits of 11111111111111111 (eighteen 1's) so that the result will be a multiple of 90 is ———

Solution Since there are eighteen 1's, no matter how we insert the $+$s, the resulting sum will be a multiple of 9. Also, the sum will be a multiple of 10, if the units digit of the sum is 0. Thus we need the number of summands to be a multiple of 10. We have 17 possible places to insert the $+$s, and we need to choose 9 of these so that we get 10 summands. Hence the number of ways is $\binom{17}{9}$.

21. The largest positive integer less than 2017 that has exactly three proper factors (a proper factor is a factor other than the number itself; for example, 11 has only one proper factor) is ———

Solution To have exactly three proper factors, the number must be either the product of two primes or the cube of a prime number. Since $11^3 = 1331$ and $13^3 = 2197$, the largest cube less than 2017 is 1331. Since $\lfloor\sqrt{2017}\rfloor = 44$, we look at primes less than 44 and consider products of two distinct primes. Of these, $5 \times 401 = 2005$ is the largest.

22. Consider the sequence $1, 3, 4, 7, 11, 18, 29, \ldots$ in which each term from the third term onwards is the sum of

the two previous terms. Of the first 100 terms of this sequence the number of terms that are multiples of 5 is

---

Solution Let us look at the sequence mod 5. This is $1,3,4,2,1,3,4,2,\ldots$ The sequence is periodic and $1,3,4,2$ repeats. Since none of these is a multiple of 5, there are no multiples of 5 in the sequence.

23. The non negative, distinct integers a,b,c,d,e form an arithmetic progression. If the sum of the numbers is 440, the maximum possible value of e is ———

Solution The progression can be assumed to be $u-2v, u-v, u, u+v, u+2v$. Since the sum of the terms is $5u = 440$ and hence $u = 88$. To make $e = u+2v$ maximum, v is maximum. Maximum v happens when $u-2v = 0$ and $v = 44$. Thus the maximum possible value of e is $88+88 = 176$.

24. Let f be defined for all positive integers as follows:

$$f(n) = \begin{cases} n^2+1 \text{ if n is odd} \\ \dfrac{n}{2} \text{ if } n \text{ is even} \end{cases}$$

The number of integers n such that $1 \leq n \leq 100$ for which $f(f(\ldots f(n))) = 1$ (where f is applied some number of times) is ———

Solution In order to get $\underbrace{f(f(\ldots f(n))\ldots)}_{n\,f's} = 1$, we need $\underbrace{f(f(\ldots f(n))\ldots)}_{n-1\,f's} = 2$. Similarly, in order to get $\underbrace{f(f(\ldots f(n))\ldots)}_{n-1\,f's} = 2$, we need $\underbrace{f(f(\ldots f(n))\ldots)}_{n-2\,f's} = 4$. So any number of the form 2^m will work. Since $2^6 = 64$ and $2^7 = 128$, the possible numbers are $2, 2^2, \ldots, 2^6$.

Also when $n = 1$, $f(1) = 2$ and $f(2) = 1$ and thus $f(f(1)) = 1$. Thus there are a total of 7 numbers.

25. M and N are three digit numbers with no digit equal to zero. If we rearrange the digits of M all the numbers obtained are less than M. N has the same digits as M but in a different order. If we rearrange the digits of N, the largest number we obtain is M. If $M + N = 1233$, M equals ———

Solution Let $M = 100a + 10b + c$, where a, b, c are digits. Then $a > b > c$, since the digits of M can not be rearranged to get a larger number. The possibilities for N are $100b + 10a + c, 100b + 10c + a, 100c + 10a + b, 100c + 10b + a, 100a + 10c + b$. We will discuss each of these cases separately:

Case 1 $N = 100b + 10a + c$. We have $M + N = 110b + 110a + 2c = 1233$, impossible since the left hand side is even and 1233 is odd.

Case 2 $N = 100b + 10c + a$. Here, $M + N = 110b + 101a + 11c = 1233$. Computing mod 11, we get $2a = 1 \pmod 1)1$ and hence $a = 6$. But then, $110b + 11c = 1233 - 606 = 627$ and $10b + c = 57$. This gives $M = 657$ but the largest number that could be obtained by rearranging digits of M is $765 > M$, a contradiction.

Case 3 $N = 100c + 10a + b$. Here, $M + N = 110a + 11b + 101c = 1233$. Computing mod 11, we get $2c = 1 \pmod 1)1$ and hence $c = 6$. This gives $110a + 11b = 627$ and $10a + b = 57$. But then $M = 576$ contradicting that the largest number obtained by rearranging the digits of M is M.

Case 4 $N = 100c + 10b + a$. Here, $M + N = 101a + 20b + 101c = 1233$. Computing mod 10, we get $a + c = 3$.

Since $a > b > c$, this is impossible.

Case 5 $N = 100a + 10c + b$. We have $M + N = 200a + 11b + 11c = 1233$. Since $b + c < 20$, $200a > 1013$ and thus $a > 5$. Also, $200a < 1233$ and hence $a < 7$. Thus $a = 6$. Now $11(b + c) = 33$ and $b + c = 3$. The only solution is $b = 2, c = 1$ and $M = 621$.

26. $ABCDEF$ is a hexagon inscribed in a circle of radius R. If $AB = CD = EF = 2$ and $BC = DE = FA = 10$, the radius R is ———

Solution Let $2\alpha, 2\beta$ be the angles subtended at the

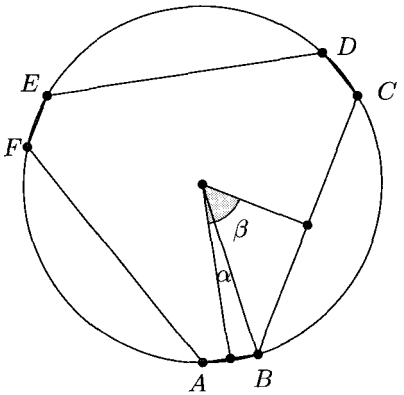

Figure 6

center of the circle by AB and BC respectively. Then $\alpha + \beta = 60°$, $\sin\alpha = \frac{1}{R}$ and $\sin\beta = \frac{5}{R}$.

$$\begin{aligned}\frac{1}{R} &= \sin\alpha = \sin(60° - \beta) \\ &= \frac{\sqrt{3}}{2}\cos\beta - \frac{1}{2}\sin\beta \\ &= \frac{1}{2}\left(\sqrt{3}\sqrt{1 - \frac{25}{R^2}} - \frac{5}{R}\right)\end{aligned}$$

Thus

$$\frac{49}{4R^2} = \frac{3}{4}\left(1 - \frac{25}{R^2}\right) \Longrightarrow 3R^2 = 124$$

and $R = 2\sqrt{\frac{31}{3}}$

27. a, b, c are real numbers satisfying the following equations:

$$\log_2(abc - 3 + \log_5 a) = 5$$
$$\log_3(abc - 3 + \log_5 b) = 4$$
$$\log_4(abc - 3 + \log_5 c) = 4$$

The value of $|\log_5 a| + \log_5 b| + |\log_5 c|$ is ———

Solution From the given equations, we have

$$abc - 3 + \log_5 a = 32$$
$$abc - 3 + \log_5 b = 81$$
$$abc - 3 + \log_5 c = 256$$

Adding, we get $3abc - 9 + \log_5(abc) = 369$ and thus $3abc + \log_5 abc = 378$. Dividing by 3, we get $abc + \log_{125} abc = 126$ and hence $abc = 125$. This gives $\log_5 a = -90$, $\log_5 b = -41$, $\log_5 c = 134$ and

$$|\log_5 a| + \log_5 b| + |\log_5 c| = 265$$

28. The largest integer such that $n^3 + 4n^2 - 15n - 18$ is a perfect cube is ———

Solution We have

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$
$$(n+2)^3 = n^3 + 6n^2 + 12n + 8$$

Since $(n+2)^3 > n^3+4n^2-15n-18$, we need only to check whether the given expression equals $(n-1)^3, n^3, (n+1)^3$

for an integral n.

If $n^3 = n^3 + 4n^2 - 15n - 18$, then $4n^2 - 15n - 18 = 0$ and this has no integer solutions.

If $(n-1)^3 = n^3 + 4n^2 - 15n - 18$, then $7n^2 - 18n - 17 = 0$ and this also has no integer solutions.

When $(n+1)^3 = n^3 + 4n^2 - 15n - 18$, we have $n^2 - 18n - 19 = 0$ and the roots are $-1, 19$. Thus the largest integer is 19.

29. In a triangle ABC, I is the incenter. The internal angle bisector of $\angle C$ meets AB at F and the circum circle of triangle ABC at Z. If $FI = 2$, $ZF = 3$ and $IC = m/n$ where m, n are relatively prime positive integers, the value of $m+n$ is ———

Solution Consider the circle AIB. Let CI intersect

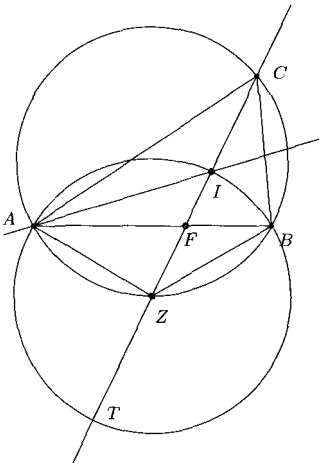

Figure 7

this circle at T. The center of this circle is Z and AB is a common chord of the circum circle and the circle AIB.

Thus

$$CF \times FZ = AF \times FB = IF \times IT$$
$$(CI + 2) \times 3 = 2 \times (3 + 5)$$

This gives $CI = \dfrac{10}{3}$ and $m + n = 13$.

30. The grid in Figure 8 shows a network of roads and pond (there are 8 horizontal lines and 8 vertical lines in the figure). You can move only horizontally or vertically from

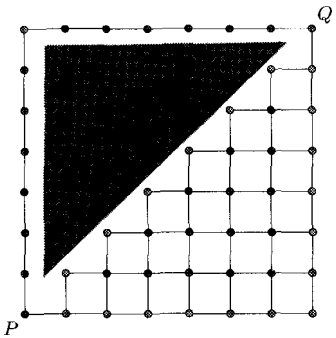

Figure 8

one grid point to an adjacent grid point. You do not know swimming and hence need to avoid going through the triangular pond at the top left corner of the grid. The number of shortest paths between P and Q is ———

Solution We count the number of ways of arriving at each grid point. Note that a grid point can be reached from a point directly next to it below or to the left. Hence the number of ways of reaching a grid point is the sum of the number of ways of reaching those points. Thus the number of ways of reaching Q is 430.

Here, we have counted the paths from P to Q that does

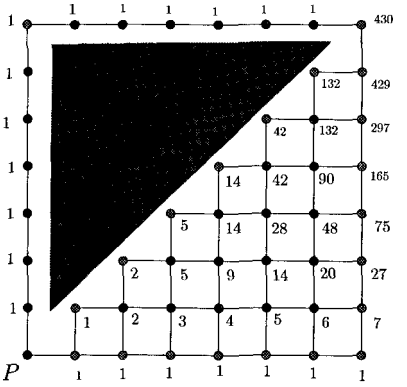

Figure 9

not cross the main diagonal and the one path through the left and top edge. In a grid of size $(n+1) \times (n+1)$ lines, the number of paths from the lower left corner to the top right corner that do not cross the diagonal is the nth Catalan number $\dfrac{1}{n+1}\binom{2n}{n}$. In the present problem, with $n = 7$, we have $\dfrac{1}{8}\binom{14}{7} = 439$.

Catalan numbers occur in many other contexts in Mathematics. One of them is discussed in the paper *S. Muralidharan, Cashier problem, The Mathematics Teacher, Vol 49, No 3&4 (2013), pp. 173 – 186.* For a more comprehensive discussion of Catalan numbers, see the book *Richard P. Stanley, Catalan Numbers, Cambridge University Press, 2015*, in which 214 different contexts in which Catalan numbers occur are listed.

FINAL – GAUSS CONTEST
NMTC at PRIMARY LEVEL
V & VI Standards

1. If
$$\frac{1}{5\frac{1}{3}}+\frac{1}{3\frac{3}{7}}+\frac{1}{4\frac{4}{7}}+\frac{1}{?}=\frac{77}{96}$$
find what should be filled in the place marked '?'

Solution

$$\frac{1}{5\frac{1}{3}}+\frac{1}{3\frac{3}{7}}+\frac{1}{4\frac{4}{7}}+\frac{1}{?}=\frac{77}{96}$$
$$\frac{1}{\frac{16}{3}}+\frac{1}{\frac{24}{7}}+\frac{1}{\frac{32}{7}}+\frac{1}{?}=\frac{77}{96}$$
$$\frac{3}{16}+\frac{7}{24}+\frac{7}{32}+\frac{1}{?}=\frac{77}{96}$$
$$\frac{18}{96}+\frac{28}{96}+\frac{21}{96}+\frac{1}{?}=\frac{77}{96}$$

Hence
$$\frac{1}{?}=\frac{77}{96}-\left(\frac{18}{96}+\frac{28}{96}+\frac{21}{96}\right)=\frac{10}{96}=\frac{1}{\frac{48}{5}}=\frac{1}{9\frac{3}{5}}$$

2. There are 10 cards numbered 1 to 10. There are three second standard children Ram, Bilal and Cynthia. The teacher selects 3 cards from the 10 cards without seeing the numbers. She distributes the cards to the children one to each. After the children note down the numbers on the cards she collects them back. Again she repeats the same process two more times. So, each child now has 3 numbers noted down. The teacher asks them to add the numbers and tell her the sums obtained by them. They told her that the sums were $10, 14, 15$. But Ram received the same card three times. Bilal and Cynthia received all cards different. What numbered cards are received by

each? Write down the steps you used to get the answer.
Solution The totals are $10, 14, 15$. Since Ram received the same card three times, his total must be a multiple of 3. The only number among the three totals that is a multiple of 3 is 15 and hence he must have received the card numbered 5 three times. Others received different cards from $1, 2, 3, 4, 6, 7, 8, 9, 10$. Among these we need to find two sets of three numbers that sum to 10 and 14 respectively. Sets of three distinct numbers that sum to 14 are $\{10, 3, 1\}, \{9, 3, 2\}, \{8, 4, 2\}, \{7, 4, 3\}, \{7, 6, 1\}$. When we choose $\{8, 4, 2\}$, from the remaining numbers, we can choose $\{1, 3, 6\}$ that sum to 10. Hence the other two students must have received the numbers $\{8, 4, 2\}$ and $\{1, 3, 6\}$.

3. In the Figure 10 $ABCD$ is a rectangle. Points P, Q, R, S are marked as in the diagram such that $AP = PQ = QB$. R is the midpoint of CD. If $AS : SD = 3 : 1$, find the ratio of the areas of triangle ASP, quadrilateral $SPRD$, triangle PQR and the trapezium $QBCR$.
Solution We can assume that $AP = PQ = QB = 2x$

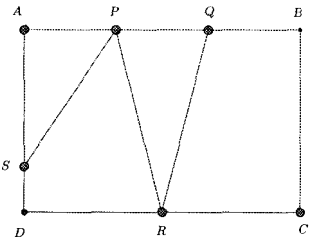

Figure 10

and $AS = 3y, SD = y$. Then $DR = RC = 3x$.
Area of the rectangle $ABCD = 6x \times 4y = 24xy$. Area of

triangle ASP is $\frac{1}{2}AP \times AS = 3xy$. Area of triangle PQR equals $\frac{1}{2}PQ \times AD = 4xy$. Also the area of trapezium $QRCB$ is $\frac{1}{2}BC \times (QB+RC) = 10xy$. Finally, area of the quadrilateral $SPRD$ equals area of rectangle $ABCD$ − area of triangle ASP − area of triangle PQR − area of trapezium $QBCR$ and hence equals $24xy - 3xy - 4xy - 10xy = 7xy$. Thus the required ratio is

$$3xy : 7xy : 4xy : 10xy = 3 : 7 : 4 : 10$$

4. Take the numbers $1, 2, 3, 4, 5, 6, 7$ and 8. We have to make two groups, A, B each containing four numbers such that

   (a) the sum of the numbers in group A is equal to the sum of the numbers in group B

   (b) Group A has a number such that when it is moved from group A to group B, the sum of the five numbers in group B is equal to twice the sum of the 3 numbers in group A.

   (c) Group B has a number such that when it moved to group A, the sum of the three numbers in Group B is $\frac{5}{7}$ of the sum of the 5 numbers in Group A.

   Find the numbers in the groups A and B.

   Solution The total of the 8 numbers is 36 and since the sum of the numbers in the groups A and B are equal, the sum of each must be 18.
   Suppose that a is moved from A to B. Then we must have $18 + a = 2(18 - a)$ and hence $a = 6$. Thus 6 must belong to group A. Let b be moved from B to A. Then $18 - b = \frac{5}{7}(18 + b)$ and $b = 3$. This gives 3 is in group B. We need to find three numbers from $1, 2, 4, 5, 7, 8$ that add up to 12 for group A. The only valid choice is

$1, 4, 7$. Thus $A = \{1, 4, 6, 7\}$ and $B = \{2, 3, 5, 8\}$.

5. Mahadevan was puzzled by the strange way in which his

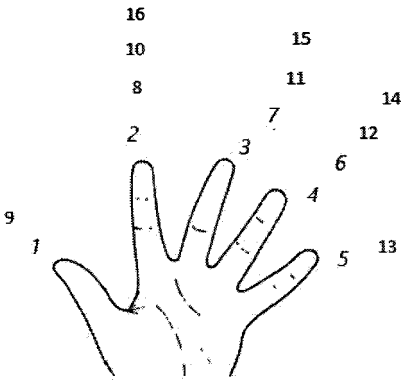

Figure 11

grand daughter was counting. She began to count on the fingers of her left hand. She started by calling the thumb 1, the first finger 2, middle finger 3, ring finger 4, little finger 5, then she reversed direction, calling the ring finger 6, middle finger 7, first finger 8, thumb 9, then back to the first finger for 10, middle finger for 11, and so on. She continued to count back and forth in this peculiar manner until she reached a count of 20 on her ring finger.

Seeing this, Mahadevan told her "If you can find on which finger you will count 2017, I will buy you an ice cream". Can you find on which finger she will count 2017? Explain the steps you used to arrive at the answer.

Solution Continuing the counting, we see that all multiples of 8 are counted on the first finger and hence

2017 will be counted on the thumb.

6. In the Figure 12, the number in each circle is the sum of the numbers in the two adjacent circles below it.

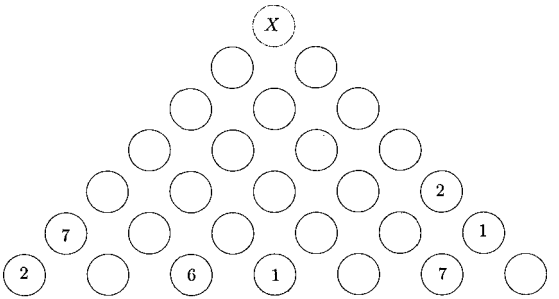

Figure 12

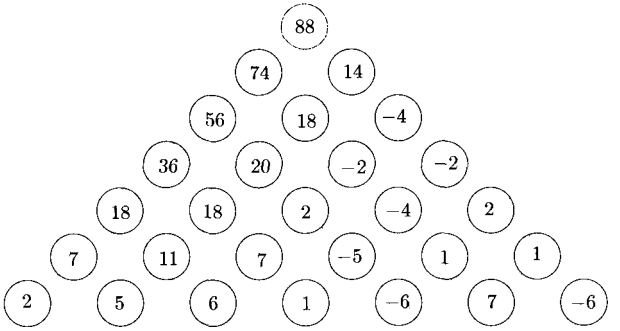

Figure 13

(a) Find X, writing the steps systematically.

(b) What is the least positive number to be added to X so that the result is a perfect square?

(c) What is the least positive number to be subtracted from X so that the result is a perfect square?

Solution Using the given numbers, we can fill up the circles working from the last row (See Figure 13). For example, since we have a 7 in the first circle of the second last row and a 2 in the first circle of the last row, the number in the second circle of the last row must be 5 to get $5 + 2 = 7$. The filled up circles are shown in Figure 13. Hence $X = 88$. The least positive number that must be added to X to make it a perfect square is 12 and the least positive number that must be subtracted from X to make it a perfect square is 7.

FINAL – KAPREKAR CONTEST
NMTC at JUNIOR LEVEL
VII & VIII Standards

1. (a) Find all three digit numbers in which ary two adjacent digits differ by 3.

   (b) There are 5 cards. Five positive integers (may be different or equal) are written on these cards, one on each card. Abhiram finds the sum of the numbers on each pair of cards. He obtains only three different totals 57, 70, 83. Find the largest integer written on a card.

Solution

(a) The digits that differ by three are the sets $\{0,3,6\},\{1,4,7\},\{2,5,8\},\{3,6,9\}$. Hence the three digit numbers whose adjacent digits differ by 3 are $630, 147, 741, 258, 852, 369, 963$.

(b) If four of the five numbers are distinct, say $a < b < c < d$, then taking sums of two numbers, we obtain the sums $a+b < a+c < a+d < c+d$. Since this gives 4 different totals, a maximum of three integers can be distinct. Let the distinct ones be a, b, c. The sums of pairs $a+b, b+c, c+a$ are distinct. Thus we need $a+b = 57, b+c = 70, c+a = 83$. This gives $a+b+c = 105$ and hence $c = 48, b = 22, a = 35$. Two other numbers must be equal to one of these. They could not be 48 (since then a sum of 96 must appear) or 22 (we will get 44 as sum). Thus the equal numbers must be 35. Thus the five numbers are $22, 35, 35, 35, 48$

2. (a) ABC is a triangle in which $AB = 24, BC = 10$ and $CA = 26$. P is a point inside the triangle. Perpendiculars are drawn to BC, AB and AC.

Length of these perpendiculars respectively are x, y and z. Find the numerical value of $5x + 12y + 13z$.

(b) If

$$\begin{aligned} x^2(y+z) &= a^2, \\ y^2(z+x) &= b^2, \\ z^2(x+y) &= c^2, \\ xyz &= abc \end{aligned}$$

prove that $a^2 + b^2 + c^2 + 2abc = 1$

Solution

(a) The areas of the triangles PBC, PAB, PAC are respectively

$$\frac{1}{2} \times BC \times x, \quad \frac{1}{2} \times AB \times y, \quad \frac{1}{2} \times AC \times z$$

and hence are equal to $5x, 12y, 13z$. Since the sum of these areas is the area of the triangle ABC, we have, using Heron's formula for the area of a triangle,

$$5x + 12y + 13z = \sqrt{30 \times 6 \times 20 \times 4} = 120$$

(b) Multiplying the first three of the given equations, we get

$$x^2y^2z^2(y+z)(z+x)(x+y) = a^2b^2c^2$$

Since $xyz = abc$, we get

$$(y+z)(z+x)(x+y) = 1$$

We have

$$\begin{aligned} a^2 + b^2 + c^2 + 2abc &= x^2(y+z) + y^2(z+x) \\ &\quad + z^2(x+y) + 2xyz \\ &= (x+y)(y+z)(z+x) \\ &= 1 \end{aligned}$$

3. If

$$X = \frac{a^2-(2b-3c)^2}{(3c+a)^2-4b^2} + \frac{4b^2-(3c-a)^2}{(a+2b)^2-9c^2} + \frac{9c^2-(a-2b)^2}{(2b+3c)^2-a^2}$$

$$Y = \frac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2} + \frac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2} + \frac{4x^2-(3y-4z)^2}{(4z+2x)^2-9y^2}$$

find $2017(X+Y)$.

Solution Consider

$$\begin{aligned}
&\frac{u^2-(v-w)^2}{(u+w)^2-v^2} + \frac{v^2-(w-u)^2}{(u+v)^2-w^2} + \frac{w^2-(u-v)^2}{(v+w)^2-u^2} \\
&\quad = \frac{(u-v+w)(u+v-w)}{(u+w+v)(u+w-v)} \\
&\qquad + \frac{(v+w-u)(v-w+u)}{(u+v+w)(u+v-w)} \\
&\qquad\quad + \frac{(w+u-v)(w-u+v)}{(v+w+u)(v+w-u)} \\
&\quad = \frac{u+v-w}{u+v+w} + \frac{v+w-u}{u+v+w} + \frac{u+w-v}{u+v+w} \\
&\quad = \frac{u+v+w}{u+v+w} = 1
\end{aligned}$$

Putting $u=a, v=2b, w=3c$, we get $X=1$. Again, putting $u=2x, v=3y, w=4z$, we get $Y=1$. Thus $2017(X+Y)=4034$.

4. The sum of the ages of a man and his wife is six times the sum of the ages of their children. Two years ago the sum of their ages was ten times the sum of the ages of their children. Six years hence the sum of their ages will be three times the sum of the ages of their children. How many children do they have?

Solution Let n be the number of children. Let x be the

sum of the ages of the man and wife and y, the sum of the ages of their children. Given $x = 6y$, $x-4 = 10(y-2n)$, and $x+12 = 3(y+6n)$. Solving, we obtain $n = 3$.

5. (a) a, b, c are three natural numbers such that $a \times b \times c = 27846$. If $\frac{a}{6} = b+4 = c-4$, find $a+b+c$.

   (b) $ABCDEFGH$ is a regular octagon with side length equal to a. Find the area of the trapezium $ABDG$.

Solution

(a) Let $\frac{a}{6} = b+4 = c-4 = t$. Then $a = 6t, b = t-4, c = t+4$ and $6t \times (t-4) \times (t+4) = 27846 = 6 \times 3 \times 7 \times 13 \times 17$ and $t(t-4)(t+4) = 17 \times 13 \times 21$. This gives $t = 17$ and hence $a+b+c = 102+13+17 = 132$.

(b) Perpendicular distance between the parallel sides AB and DG $a+a\cos 45°$ and $GD = a+2a\cos 45°$. Thus the area is

$$\frac{1}{2}\left(a+\frac{a}{\sqrt{2}}\right)\left(a+a\sqrt{2}\right) = \frac{a^2(3+2\sqrt{2})}{2\sqrt{2}}$$

6. (a) If a, b, c are positive real numbers such that no two of them are equal, show that

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b)$$

is always positive.

(b) In the figure below, P, Q, R, S are points on the sides of the triangle ABC such that

$$CP = PQ = QB = BA = AR = RS = SC$$

Find the $\angle C$.

Solution

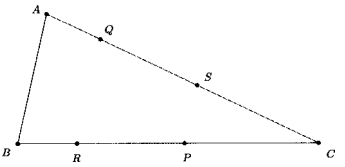

Figure 14

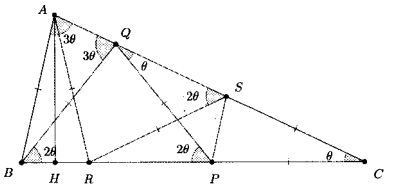

Figure 15

(a) Due to the symmetry of the expression, we can assume that $a > b > c$. We have $a - c > b - c$ and hence $a(a-b)(a-c) > b(a-b)(b-c)$. Also, $c-a < 0$ and $c-b < 0$ and hence $c(c-a)(c-b) > 0$. Thus

$$a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) > 0$$

(b) In Figure 15, let $\angle SCP = \theta$. Let AH be perpendicular to BC from A. Using the equality of the line segments, we obtain $\angle AQB = 3\theta = \angle BAQ$ and $\angle RAQ = 2\theta$. Thus $\angle BAR = \theta$ and $\angle HAR = \dfrac{\theta}{2}$. In the right angled triangle AHC, we have $\angle QAH + \angle QCH = \dfrac{\pi}{2}$. But $\angle QAH = \angle QAB - \angle HAR = 3\theta - \dfrac{\theta}{2} = \dfrac{5\theta}{2}$. Hence $\dfrac{5\theta}{2} + \theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{\pi}{7}$

FINAL – BHASKARA CONTEST
NMTC at JUNIOR LEVEL
IX & X Standards

1. (a) Find all prime numbers p such that $4p^2 + 1$ and $6p^2 + 1$ are also primes.

   (b) Determine real numbers x, y, z, u such that

$$xyz + xy + yz + zx + x + y + z = 7$$
$$yzu + yz + zu + uy + y + z + u = 9$$
$$zux + zu + ux + xz + z + u + x = 9$$
$$uxy + ux + xy + yu + u + x + y = 9$$

Solution

(a) When $p = 5$, $4p^2 + 1 = 101$ and $6p^2 + 1 = 151$ and both of these are primes. We will show that no other primes have this property.

When $p \neq 5$, the following table gives mod 5 of $4p^2 + 1$ and $6p^2 + 1$

p mod 5	$(4p^2 + 1)$ mod 5	$(6p^2 + 1)$ mod 5
1	0	2
2	2	0
3	2	0
4	0	2

From the table, it is clear that when $p \neq 5$, either $4p^2 + 1$ or $6p^2 + 1$ is a multiple of 5.

(b) Adding 1 to both sides of the given equations, we obtain

$$(1 + x)(1 + y)(1 + z) = 8$$
$$(1 + y)(1 + z)(1 + u) = 10$$
$$(1 + z)(1 + u)(1 + x) = 10$$
$$(1 + u)(1 + x)(1 + y) = 10$$

Multiplying the above equations, we get

$$(1+x)^3(1+y)^3(1+z)^3(1+u)^3 = 8 \cdot 10^3$$

and hence

$$(1+x)(1+y)(1+z)(1+u) = 20$$

This gives $1+u = \dfrac{20}{8} = \dfrac{5}{2}$ and hence $u = \dfrac{3}{2}$. Similarly, we obtain $x = y = z = 1$.

2. If x, y, z, p, q, r are distinct real numbers such that

$$\begin{aligned}
\frac{1}{x+p} + \frac{1}{y+p} + \frac{1}{z+p} &= \frac{1}{p} \\
\frac{1}{x+q} + \frac{1}{y+q} + \frac{1}{z+q} &= \frac{1}{q} \\
\frac{1}{x+r} + \frac{1}{y+r} + \frac{1}{z+r} &= \frac{1}{r}
\end{aligned}$$

find the numerical value of $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r}$.

Solution Consider the cubic equation in t

$$\frac{1}{x+t} + \frac{1}{y+t} + \frac{1}{z+t} = \frac{1}{t}$$

Given that p, q, r are the roots of this equation. The above equation can be written as

$$2t^3 + t^2(x+y+z) + xyz = 0$$

Hence $pq+qr+rp$ is given by the coefficient of t divided by the coefficient of t^3 and hence is 0. Also $pqr = -\dfrac{xyz}{2}$. Thus

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{pq+qr+rp}{pqr} = 0$$

3. ADC and ABC are triangles such that $AD = DC$ and $CA = AB$. If $\angle CAB = 20°$ and $\angle ADC = 100°$, without using Trigonometry, prove that $AB = BC+CD$.
Solution Draw the circle through A, D, C. Let this meet AB at E. We have $\angle DAE = 20° = \angle DCE$ and $\angle CDE = \angle CAE = 20°$. Thus $CE = DE$. Also,

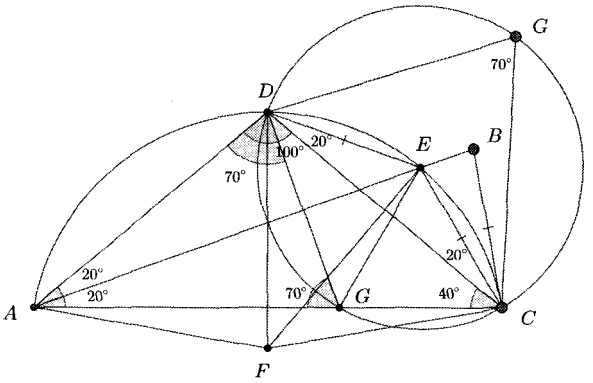

Figure 16

$\angle ABC = 70°$ and $\angle AEC = 180° - \angle EAC - \angle ECA = 110°$. Thus $\angle BEC = 70°$. It follows that $CB = CE$. Let the circle with center E and radius EC cut AC at G. Since $\angle DEC = 140°$, it follows that $\angle DGA = 70°$. It follows that $\angle ADG = 70°$ and hence $AD = AG$. Thus $AB = AG + GC = AD + EC = CD + BC$.

4. (a) a, b, c, d are positive real numbers such that $abcd = 1$. Prove that

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{1+da}{1+d} \geq 4$$

(b) In a scalene triangle ABC, $\angle BAC = 120°$. The bisectors of the angles A, B and C meet the opposite sides in P, Q and R respectively. Prove that the circle on QR as diameter passes through the point P.

Solution

(a) We can find positive real numbers x, y, z, w such that

$$a = \frac{y}{x}, \quad b = \frac{z}{y}, \quad c = \frac{t}{z}, \quad d = \frac{x}{t}$$

The given inequality is equivalent to proving

$$S = \frac{x+z}{x+y} + \frac{y+t}{y+z} + \frac{z+x}{z+t} + \frac{t+y}{t+x} \geq 4$$

We first observe the following:
For $a, b > 0$,

$$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$$

Since $(a-b)^2 \geq 0$, we have $a^2+b^2 \geq 2ab$ and hence $a^2 + b^2 + 2ab \geq 4ab$. Thus $(a+b)^2 \geq 4ab$. This gives

$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} \geq \frac{4}{a+b}$$

Now,

$$\begin{aligned} S &= (x+z)\left(\frac{1}{x+y} + \frac{1}{z+t}\right) + (y+t)\left(\frac{1}{y+z} + \frac{1}{t+x}\right) \\ &\geq (x+z)\left(\frac{4}{x+y+z+t}\right) + (y+t)\left(\frac{4}{x+y+z+t}\right) \\ &= 4 \end{aligned}$$

This completes the proof.

(b) We first find the length of the bisector AP.

The area of the triangle ABC is $\frac{1}{2}bc\sin A$. Again, it is also equal to the sum of the areas of the triangles APB and APC. Thus

$$\frac{1}{2}bc\sin A = \frac{1}{2}AP(b+c)\sin\frac{A}{2}$$

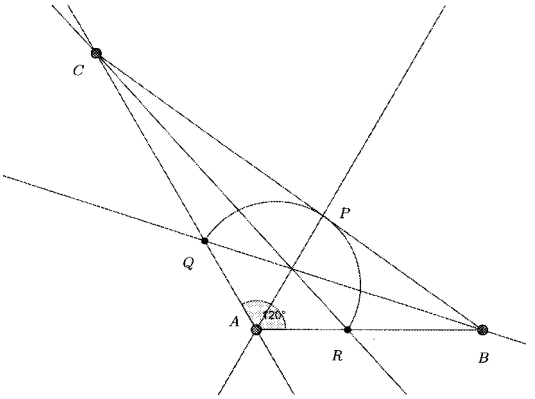

Figure 17

and hence $AP = 2\dfrac{bc}{b+c}\cos\dfrac{A}{2} = \dfrac{bc}{b+c}$.
Since P divides BC in the ratio $c : b$, $BP = \dfrac{ac}{b+c}$. Now,

$$AP : BP = \frac{bc}{b+c} : \frac{ac}{b+c} = b : a = AR : RB$$

Thus PR bisects the $\angle APB$. Similarly, PQ bisects the $\angle APC$. Hence $\angle QPR = 90°$ and P lies on the semi circle with QR as diameter.

5. (a) Prove that $x^4 + 3x^3 + 6x^2 + 9x + 12$ can not be expressed as a product of two polynomials of degree 2 with integer coefficients.

   (b) $2n + 1$ segments are marked on a line. Each of these segments intersects at least n other segments. Prove that one of these segments intersects all other segments.

Solution

(a) Suppose that

$$x^4+3x^3+6x^2+9x+12=(x^2+ax+b)(x^2+cx+d)$$

where a,b,c,d are integers. Comparing coefficients of powers of x, we get

$$a+c=3,\quad b+d+ac=6,\quad bc+ad=9,\quad bd=12$$

Since $bd=12$, exactly one of b,d is a multiple of 3. Suppose that $3 \mid b$ and $3 \nmid d$. Now, from $bc+ad=9$, we get $3 \mid a$. But $b+d+ac=6$ and $3 \mid b, 3 \mid ac$ giving $3 \mid d$, a contradiction. We obtain a contradiction similarly if we assume $3 \nmid b$ and $3 \mid d$. Thus the given quartic can not be written as a product of two quadratic polynomials with integer coefficients.

(b) Mark the segments on a horizontal number line with the positive direction on the right, so that all endpoints are assigned a coordinate. Of the finitely many segments, some segment L has a left endpoint whose coordinate is maximal; similarly, some segment R has a right endpoint whose coordinate is minimal. L intersects at least $n+1$ of the given segments (including L itself), as does R, giving a total count of $2n+2>2n+1$ segments. Hence, some given segment S intersects both L and R. If any given segment S' lies entirely to the right of S; then its left endpoint is to the right of S. But the left endpoint of L cannot lie to the right of S because $L \cap S \neq \phi$. Thus, S' has a left endpoint farther to the right than that of L, a contradiction. Similarly, no given segment can lie entirely to the left of S. It follows that any given segment intersects S, as required.

6. If a, b, c, d are positive real numbers such that $a^2 + b^2 = c^2 + d^2$ and $a^2 + d^2 - ad = b^2 + c^2 + bc$, find the value of $\dfrac{ab + cd}{ad + bc}$.

Solution Consider a cyclic quadrilateral $ABCD$ with

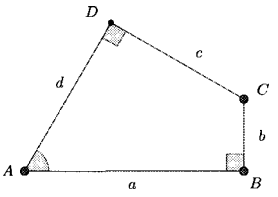

Figure 18

$AB = a, BC = b, CD = c, AD = d$ and $\angle BAD = 60°, \angle ABC = 90°, \angle ADC = 90°$. The given conditions on a, b, c, d ensure that such a quadrilateral exists. Now, area of the quadrilateral is given by $\dfrac{1}{2}(ab + cd)$ as well as $\dfrac{1}{2}(ad + bc)\sin 60°$. Thus

$$\frac{ab + cd}{ad + bc} = \sin 60° = \frac{\sqrt{3}}{2}$$

Solution 2 If $a = c$, then $b = d$ and from $a^2 + d^2 - ad = b^2 + c^2 + bc$ we get $2ab = 0$, a contradiction. Thus we can assume that $a \neq c$. From $a^2 + d^2 - ad = b^2 + c^2 + bc$ and $a^2 + b^2 = c^2 + d^2$, we get $ad + bc = a^2 + d^2 - b^2 - c^2 =$

$2(a^2 - c^2)$.

$$\begin{aligned}
&\left(\frac{ab+cd}{ad+bc}\right)^2 - 1 \\
&= \frac{(a^2b^2 + c^2d^2 + 2abcd) - (a^2d^2 + b^2c^2 + 2abcd)}{(ad+bc)^2} \\
&= \frac{a^2(b^2 - d^2) + c^2(d^2 - b^2)}{4(a^2 - c^2)^2} \\
&= \frac{a^2(c^2 - a^2) + c^2(a^2 - c^2)}{4(a^2 - c^2)^2} \\
&= -\frac{(a^2 - c^2)^2}{4(a^2 - c^2)^2} \\
&= -\frac{1}{4}
\end{aligned}$$

Since a, b, c, d are positive,

$$\frac{ab+cd}{ad+bc} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

FINAL – RAMANUJAN CONTEST
NMTC at INTER LEVEL
XI & XII Standards

1. $f(n)$ is defined for all integers $n \geq 0$. If $f(1) = 1$ and f satisfies

$$f(m-n) + f(m+n) = \frac{f(2m) + f(2n)}{2}$$

for all integers $m \geq n \geq 0$, find $f(2017)$.

Solution Putting $m = 0, n = 0$, we get $f(0) + f(0) = f(0)$ and hence $f(0) = 0$. Putting $n = 0$, we get

$$f(m-0) + f(m+0) = \frac{f(2m) + f(0)}{2}$$

and hence $f(2m) = 4f(m)$ for all m. This gives $f(2) = 4f(1) = 4$. We prove $f(m) = m^2$. Clearly, this holds for $m = 1$. Assume that $f(k) = k^2$ for all $k \leq m$.

$$\begin{aligned} f(m-1) + f(m+1) &= \frac{1}{2}(f(2m) + f(2)) \\ &= \frac{1}{2}(4f(m) + 4) \\ &= \frac{1}{2}(4m^2 + 4) \\ &= 2(m^2 + 1) \end{aligned}$$

Hence

$$\begin{aligned} f(m+1) &= 2(m^2+1) - f(m-1) = 2(m^2+1) - (m-1)^2 \\ &= m^2 + 2m + 1 \\ &= (m+1)^2 \end{aligned}$$

completing the induction.

Thus $f(2017) = 2017^2$.

2. Let ABC be a triangle. Squares are described as shown in the Figure 19. If S_i, $i = 1, 2, \ldots, 6$ denote their areas, find $\dfrac{S_4 + S_5 + S_6}{S_1 + S_2 + S_3}$.

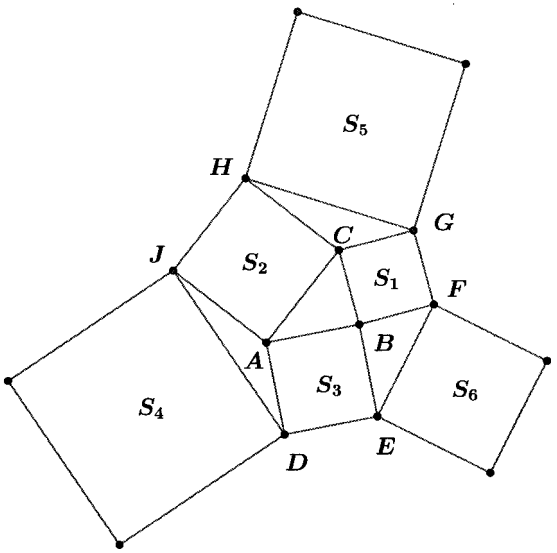

Figure 19

Solution Noting that $\angle JAD = 180° - A$, we have

$$JD^2 = c^2 + b^2 + 2bc\cos A$$

Similarly,

$$EF^2 = a^2 + c^2 + 2ac\cos B$$
$$GH^2 = a^2 + b^2 + 2ab\cos C$$

Hence

$$\begin{aligned} S_4 + S_5 + S_6 &= 2(a^2 + b^2 + c^2) \\ &\quad + 2(ab\cos C + bc\cos A + ca\cos B) \\ &= 2(a^2 + b^2 + c^2) + (a^2 + b^2 - c^2) \\ &\quad + (c^2 + a^2 - b^2) + (b^2 + c^2 - a^2) \\ &= 3(a^2 + b^2 + c^2) \end{aligned}$$

Since $S_1 + S_2 + S_3 = a^2 + b^2 + c^2$, it follows that $\dfrac{S_4 + S_5 + S_6}{S_1 + S_2 + S_3} = 3$.

3. The four digit number 1805 has the curious property that $\dfrac{1805}{05} = 19^2$, that is, when the number is divided by its last two digits, the quotient is the square of one more than its first two digits. Find all four digit numbers with this property.

   Solution Let the four digit number be $100x + y$ where $10 \le x \le 99$ and $1 \le y \le 99$. We need

   $$100x + y = (x + 1)^2 y = x^2 y + 2xy + y$$

   and hence $100x = xy(x + 2)$. This gives $y = \dfrac{100}{x + 2}$. Thus $10 \le x \le 99$ and $x + 2$ must divide 100. The possible values of x are $18, 23, 48, 98$. Thus the numbers are $1805, 2304, 4802, 9801$.

4. Find all polynomial functions $f(x)$ with real coefficients such that for all a, b, c real, f satisfies

   $$\begin{aligned} f(a + b - 2c) + f(b + c - 2a) + f(c + a - 2b) \\ = 3f(a - b) + 3f(b - c) + 3f(c - a) \end{aligned}$$

   Solution When $a = b = c = 0$, we get $3f(0) = 9f(0)$ and hence $f(0) = 0$.

Put $x = b - c, y = c - a, z = a - b$. Then, for all x, y, z such that $x + y + z = 0$, we have

$$f(x - y) + f(y - z) + f(z - x) = 3(f(x) + f(y) + f(z))$$

In particular, letting $z = 0, y = -x$, we have

$$f(2x)+2f(-x) = 3(f(x)+f(-x)) \Longrightarrow f(2x) = 3f(x)+f(-x)$$

If $f(x) = \sum a_n x^n$, we must have

$$\sum 2^n a_n x^n = 3\sum a_n x^n + \sum (-1)^n a_n x^n$$

Equating the coefficient of x^n, we get

$$(2^n - 3 - (-1)^n)\, a_n = 0$$

This gives $a_n = 0$ for $n \geq 3$. Hence $f(x) = Ax^2 + Bx$ where a, b are real constants.
Since

$$\begin{aligned}(a + b - 2c)^2 + (c + a - 2b)^2 + (b + c - 2a)^2 \\ = 3(a - b)^2 + 3(b - c)^2 + 3(c - a)^2\end{aligned}$$

and

$$\begin{aligned}(a + b - 2c) + (b + c - 2a) + (c + a - 2b) = 0 \\ = 3(a - b) + 3(b - c) + 3(c - a)\end{aligned}$$

it follows that the function $f(x) = Ax^2 + Bx$ indeed has the stated property.

5. The area of a convex quadrilateral $ABCD$ is one. Find the minimum value of

$$AB + BC + CD + DA + AC + BD$$

Solution If θ is the angle of intersection of the diagonals $ABCD$, then the area is given by $\frac{1}{2}AC \cdot BD \sin\theta$ and hence we have $AC \cdot BD = \dfrac{2}{\sin\theta} \geq 2$. Now,

$$\frac{AC + BD}{2} \geq \sqrt{AC \cdot BD} \geq \sqrt{2}$$

and hence $AC + BD \geq 2\sqrt{2}$.
Also, area of the quadrilateral is given by

$$\frac{1}{2}AB \cdot AD \sin\angle DAB + \frac{1}{2}CB \cdot CD \sin\angle BCD = 1$$
$$\frac{1}{2}BA \cdot BC \sin\angle ABC + \frac{1}{2}DA \cdot DC \sin\angle ADC = 1$$

Thus

$$\begin{aligned} AB \cdot AD + CB \cdot CD &\geq AB \cdot AD \sin\angle DAB \\ &\quad + \frac{1}{2}CB \cdot CD \sin\angle BCD \\ &= 2 \end{aligned}$$

Hence $AB \cdot AD + CB \cdot CD \geq 2$ and similarly,

$$BA \cdot BC + DA \cdot DC \geq 2$$

Combining, we obtain

$$(AB + CD)(AD + BC) \geq 4$$

and hence

$$\frac{AB + BC + CD + DA}{2} \geq \sqrt{(AB + CD)(AD + BC)} \geq 2$$

Thus

$$AB + BC + CD + DA + AC + BD \geq 4 + 2\sqrt{2}$$

Note that equality is attained for a square with side 1.

6. A basket player player during a practice session shoots basketball 100 times. First time he scores a point and second time he misses it. For the following shots, the probability of him scoring a point is equal to number of points scored before this shot divided by number of shots taken before this shot; for example, if he is into his 21st shot and he has scored 13 points in the first 20 shots then the probability of him scoring in 21st shot is $\frac{13}{20}$. What is the probability that

   (a) he scores 33 points in the 100 shots?

   (b) he scores 67 points in the 100 shots?

   Solution Whatever is the score, the probability is $\frac{1}{99}$. This can be seen as follows:

   If the score is k, all numbers from 1 to $k-1$ appear in the numerator. Now consider the non scoring shots. Suppose that the second non scoring shot occurs as the pth shot. This means, up to $p-1$ shots, the score is $p-2$ and the pth shot is non scoring. Thus the probability is $\frac{(p-1)-(p-2)}{p-1} = \frac{1}{p-1}$ and this contributes the integer 1 to the numerator. Now consider the third non scoring shot. If this is the qth shot, up to $q-1$ shots, we have $q-3$ scoring shots and hence this shot contributes the number 2 to the numerator. Thus at the mth non scoring shot, we get the number $m-1$. Hence the numerator is $(k-1)!(99-k)!$. Thus the probability is

$$\frac{\binom{98}{k-1}(k-1)!(99-k)!}{99!} = \frac{1}{99}$$

7. Find all integers x, y such that $x^6 + 3x^3 + 1 = y^4$.
   Solution Put $t = x^3$, we get $t^2 + 3t + 1 - y^4 = 0$ and hence $9 - 4(1 - y^4) = n^2$ and hence $n^2 - 4y^4 = 5$ and

$(n-2y^2)(n+2y^2) = 5$. There are only 4 cases to consider.

$$\begin{aligned} n - 2y^2 &= 1, \quad n + 2y^2 = 5, \\ n - 2y^2 &= 5, \quad n + 2y^2 = 1, \\ n - 2y^2 &= -1, \quad n + 2y^2 = -5, \\ n - 2y^2 &= -5, \quad n + 2y^2 = -1 \end{aligned}$$

It is easy to see that $x = 0, y = \pm 1$ are the only integer solutions.

8. The positive integers from 1 to n inclusive are written on a blackboard. After one number is erased, the average (arithmetic mean) of the remaining $n - 1$ numbers is $48\frac{41}{48}$. Determine all possible values of n and the number that was erased.

Solution We will prove that $n = 93$ and the number erased is 59.

When we erase one number from $1, 2, \ldots, n$, the maximum sum of the remaining numbers is

$$\frac{n(n+1)}{2} - 1 = \frac{n^2 + n - 2}{2} = \frac{(n+2)(n-1)}{2}$$

and the minimum sum is $\dfrac{(n-1)n}{2}$. Hence the average of the remaining numbers lies between $\dfrac{n+2}{2}$ and $\dfrac{n}{2}$. Hence we have

$$\frac{n}{2} \leq 48\frac{41}{48} \leq \frac{n+2}{2} \Longrightarrow 96 \leq n \leq 97$$

Also, if the number erased is k, we have

$$\frac{\frac{n(n+1)}{2} - k}{n-1} = 48\frac{41}{48} \Rightarrow \frac{n(n+1)}{2} - k = (n-1)\frac{48 \cdot 48 + 41}{48}$$

Since $48 \cdot 48 + 41$ is prime to 48, it follows that 48 divides $n - 1$. Since $96 \leq n \leq 97$, and $n \equiv 1 \pmod{48}$, it

follows that $n = 97$. Now, if k is the number erased, we have

$$\frac{\frac{97 \cdot 98}{2} - k}{96} = 48\frac{41}{48}$$

Solving for k we get $k = 63$.

ARYABHATTA CONTEST
SENIOR

1. Let $S = \{1, 2, 3, \ldots, 15\}$. A is a subset of S containing at least 8 numbers. Show that there are two numbers $a, b \in A$ such that either $|a - b| = 4$ or $|a - b| = 11$.
Solution Place the numbers in order around a circle and

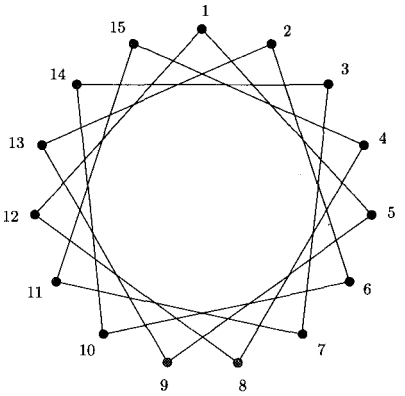

Figure 20

connect two of the points if they differ by 4 or 11. The resulting graph has degree 2 and has 15 edges. If we could select 8 vertices no two of which are connected, then when we remove those vertices and the edges incident on them, we should be removing 16 edges, a contradiction, since the graph has only 15 edges.

2. A student writes 4 exams with 50 marks each. In how many ways can he score exactly 100 marks? In how many ways can he score greater than 100 marks?
Solution The number of ways in which the student can score N marks is the coefficient of x^N in the expansion

$(1 + x + \cdots + x^{50})^4$.

$$(1 + x + \cdots + x^{50})^4 = \left(\frac{1 - x^{51}}{1 - x}\right)^4$$
$$= (1 - 4x^{51} + 6X^{102} - 4X^{153} + X^{204})$$
$$\times \left(1 + 4x + \frac{4 \cdot 5}{1 \cdot 2}x^2 + \cdots + \binom{n+3}{3}x^n + \cdots\right)$$

Hence the number of ways in which he can get exactly 100 is

$$\binom{103}{3} - 4\binom{52}{3} = 88451$$

For each way of scoring $k < 100$, there is a way of scoring $200 - k$ by replacing each right answer by wrong answer and wrong answer by right answer. Thus the number of ways of scoring < 100 is equal to number of ways of scoring > 100 and hence equals $51^4 - 88451$.

3. Let $f : \{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\}$ be a function such that $f(0) = 0, f(1) = 1$ and

$$f\left(m^2 + n^2\right) = f(m)^2 + f(n)^2$$

for all non negative integers m, n. Find $f(7)$.

Solution We will show that $f(N) = N$ for all N.

We have $f(2) = f(1^2 + 1^2) = f(1)^2 + f(1)^2 = 2$ and, we can easily see that $f(4) = 4$ and $f(5) = 5$ because $5 = 2^2 + 1^2$.

To see $f(3) = 3$, note that

$$25 = f(5^2) = f(3)^2 + f(4)^2 \implies f(3) = 3$$

Suppose you already know that $f(n) = n$ for all $n < N$. We want to find a, b, c, all smaller than N, such that $N^2 + a^2 = b^2 + c^2$. If we find such a triplet, we immediately conclude that $f(N) = N$, and the induction is complete.

For odd $N > 6$, we take

$$N^2 + \left(\frac{N-5}{2}\right)^2 = (N-2)^2 + \left(\frac{N+3}{2}\right)^2$$

For even $N > 6$, we take

$$N^2 + \left(\frac{N}{2} - 5\right)^2 = (N-4)^2 + \left(\frac{N}{2} + 3\right)^2$$

Since $f(n) = n$ for small values of n, this inductive step finishes the argument.

4. Let a, b, c be real numbers different from $\pm\frac{1}{\sqrt{3}}$. Let

$$A = \frac{3a - a^3}{3a^2 - 1}, \quad B = \frac{3b - b^3}{3b^2 - 1}, \quad C = \frac{3c - c^3}{3c^2 - 1}$$

If $a + b + c = abc$, show that $A + B + C = ABC$

Solution Let $\alpha, \beta, \gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $a = \tan\alpha, b = \tan\beta, c = \tan\gamma$. From

$$\tan(\alpha + \beta + \gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 + \tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha}$$

it follows that $a+b+c = abc$ if and only if $\alpha+\beta+\gamma = k\pi$. In this case, we have $3\alpha + 3\beta + 3\gamma = 3k\pi$ and hence

$$\tan 3\alpha + \tan 3\beta + \tan 3\gamma = \tan 3\alpha \tan 3\beta \tan 3\gamma$$

Since $\tan 3\alpha = \dfrac{3a - a^3}{1 - 3a^2}$, and similar expressions for $\tan 3\beta$ and $\tan 3\gamma$, the result follows.

5. A rectangular billiard table has size $139\,cm \times 181\,cm$ and has pockets in its four corners. A ball is hit from the lower left corner of the table at an angle $45°$ to the horizontal. All bounces obey the law of reflection – the angle of

incidence equals the angle of reflection (The initial path of the ball is shown). Find the number of times it bounces off the walls of the table and the pocket it lands at the end.

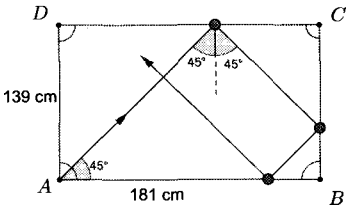

Figure 21

Solution Suppose that the length and breadth of the

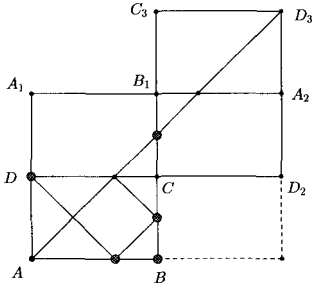

Figure 22

table are m, n respectively. To find the reflected path, we can reflect the entire board on the wall where the ball hits and continue the path of the ball to the reflected board, see Figure 22.

When the ball lands in a pocket, we could have made $(a-1)$ horizontal reflections and $(b-1)$ vertical reflections.

This is placing a copies of the board horizontally and b copies of the board vertically. Then, since the angle of projection is $45°$, we need $ma = nb$. In this case, we need integers a, b such that $181a = 139b$. Since 189 and 131 have no common divisors, it follows that $a = 139$ and $b = 181$ and the number of bounces is $a + b = 189 + 131 = 320$.

It is easy to see that the ball lands in C.

6. Let x, y, z be real numbers. Find the minimum value of the expression

$$\sqrt{x^2+4} + \sqrt{(y-x)^2+25} + \sqrt{(z-y)^2+64} + \sqrt{(14-z)^2+100}$$

Find also one triple (x, y, z) for which the expression above attains that minimum value.

Solution Consider the points $A(0,0), B(2,x), C(7,y)$, $D(15,z)$, and $E(25,14)$. The given expression equals $AB + BC + CD + DE$ and hence the minimum distance is $AE = \sqrt{25^2+14^2} = \sqrt{821}$.

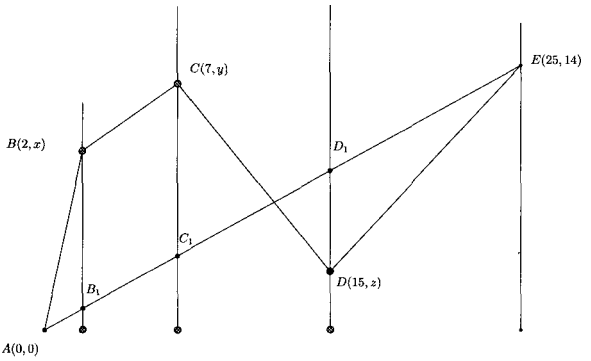

Figure 23

The minimum distance is attained when the points are at

the intersection of the line AE with the ordinates at $2, 7$ and 15. These are $x = 28/25, y = 91/25, z = 42/5$.

7. A city has a circular wall around it and the wall has four gates pointing north, south, east and west. A house stands outside the city, 16 km north of the north gate, and it can be just seen appearing on the horizon from a point 48 km east of the south gate. What is the diameter of the circular wall that surrounds the city?

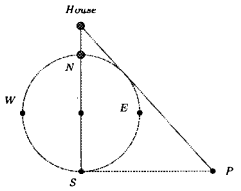

Figure 24

Solution Let r be the radius of the circular wall. From

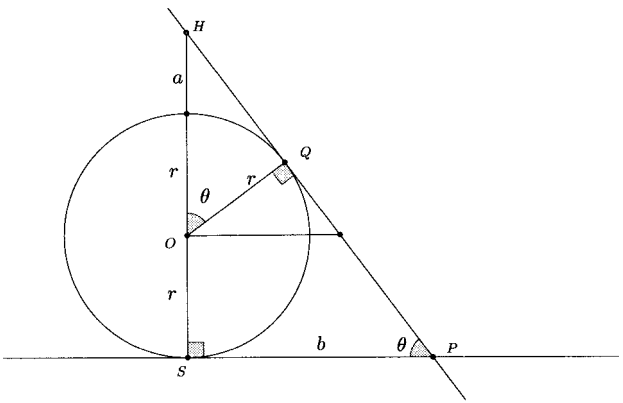

Figure 25

Figure 25, using tangent property, we have $a(a + 2r) =$

HQ^2. Also, note that $PS = PQ = b$. Hence from right angled triangle HSP, we have

$$(b + \sqrt{a(a+2r)})^2 = (a+2r)^2 + b^2$$

Substituting $a = 16$ and $b = 48$ in the above equation, we obtain $2r = 48$ km.

8. Find all values of the real number c such that the set of solutions (x, y) of the system

$$\frac{x^2 + y^2 - 16x + 10y + 65}{x^2 + y^2 - 14x + 12y + 79} \leq 0$$
$$(x - c)(y + c) = 0$$

forms a line segment.

Solution The numerator $x^2 + y^2 - 16x + 10y + 65 = 0$

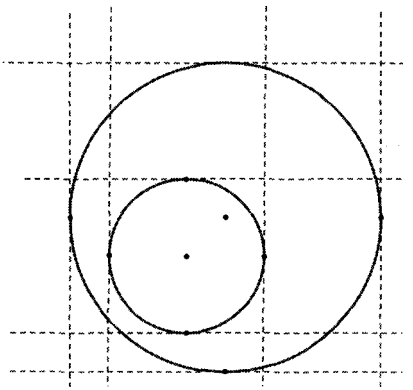

Figure 26

is a circle C_1 with center $(8, -5)$ and the denominator $x^2 + y^2 - 14x + 12y + 79 = 0$ is a circle C_2 with center $(7, -6)$. The ratio is negative at a point (x, y) if the point lies inside C_1 but outside C_2. Consider the tangents $x = c_1, x = c_2, x = c_3, x = c_4$, where $c_1 < c_2 < c_3 < c_4$, to these circles shown in Figure 26.

When $c \in (c_1, c_2) \cup (c_3, c_4)$, the vertical line $x = c$ meets the region between the circles in a line segment. Similarly, when $d \in (d_4, d_3) \cup (d_2, d_1)$, where $y = d_i$ are tangents to the circles, the horizontal line $y = d$ meets the region between the circles in a line segment.

The vertical tangents to these circles are $x = 8 \pm 2\sqrt{6}$ and $x = 7 \pm \sqrt{6}$. The horizontal tangents are $y = -5 \pm 2\sqrt{6}$ and $y = -6 \pm \sqrt{6}$. Hence the values of c for which the condition given is satisfied is given by

$$c \in \left(8 - 2\sqrt{6}, 7 - \sqrt{6}\right) \cup \left(7 + \sqrt{6}, 8 + 2\sqrt{6}\right) \cup \left(-6 + \sqrt{6}, -5 + 2\sqrt{6}\right) \cup \left(-5 - 2\sqrt{6}, -6 - \sqrt{6}\right)$$

9. a, b are real numbers. Given that the equation

$$x^4 + ax^3 + 2x^2 + bx + 1 = 0$$

has a real root, show that $a^2 + b^2 \geq 8$

Solution 1 Let r be a real root of the equation. We can write the equation as

$$ar^2 + b = -\frac{(r^2+1)^2}{r}$$

By Cauchy-Schwartz inequality,

$$(r^4 + 1)(a^2 + b^2) \geq (ar^2 + b)^2$$

Hence

$$\begin{aligned} a^2 + b^2 &\geq \frac{(ar^2+b)^2}{r^4+1} = \frac{(r^2+1)^4}{r^2(r^4+1)} \\ &= \frac{r^8 + 2r^4 + 1 + 4r^4 + 4r^6 + 4r^2}{r^2(r^4+1)} \\ &= \frac{r^4+1}{r^2} + \frac{4r^2}{r^4+1} + 4 \\ &\geq 4 + 4 \text{ by AM - GM inequality} \\ &= 8 \end{aligned}$$

Solution 2 If r is a real root, then $r \neq 0$ and

$$r^2 + ar + 2 + \frac{b}{r} + \frac{1}{r^2} = 0$$

Hence

$$\left(r + \frac{a}{2}\right)^2 + \left(\frac{1}{r} + \frac{b}{2}\right)^2 = \frac{a^2}{4} + \frac{b^2}{4} - 2$$

Since the left hand side is non negative, we get

$$\frac{a^2}{4} + \frac{b^2}{4} - 2 \geq 0$$

and hence $a^2 + b^2 \geq 8$.

10. A basket player player during a practice session shoots basketball 100 times. First time he scores a point and second time he misses it. For the following shots, the probability of him scoring a point is equal to number of points scored before this shot divided by number of shots taken before this shot; for example, if he is into his 21st shot and he has scored 13 points in the first 20 shots then the probability of him scoring in 21st shot is $\frac{13}{20}$. What is the probability that

    (a) he scores 45 points in the 100 shots?

    (b) he scores 55 points in the 100 shots?

Solution See Solution to Problem 6, Inter Final.

Pre RMO 2017

1. How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3?
Solution A number is divisible by 3 if the sum of its digits is divisible by 3. Hence we need to count the number of three digit numbers whose sum of digits is divisible by 7 and 3 and hence by 21. The maximum sum of digits of a 3 digit number is 27 and hence we need the numbers $100a + 10b + c$ where $a + b + c = 21$. If $a = 0$ then $b + c = 21$ and this is impossible since b, c are digits. Similarly, $a = 1, 2$ are also not possible. Let $a = 3$. Then $b + c = 18$ and this has 1 solution $b = c = 9$. Similarly, the number of solutions for $a = 4, 5, \ldots, 9$ are respectively $2, 3, 4, 5, 6, 7$. Hence the number of integers satisfying the given property is $1 + 2 + \cdots + 7 = 28$.

2. Suppose a, b are positive real numbers such that $a\sqrt{a} + b\sqrt{b} = 183$, $a\sqrt{b} + b\sqrt{a} = 182$. Find $\frac{9}{5}(a + b)$.
Solution Let $\sqrt{a}, \sqrt{b}$ be roots of the quadratic $X^2 - tX + s = 0$. Then

$$\begin{aligned} a\sqrt{a} + b\sqrt{b} &= (\sqrt{a} + \sqrt{b})^3 - 3\sqrt{ab}(a + b) \\ &= t^3 - 3ts = 183 \\ a\sqrt{b} + b\sqrt{a} &= \sqrt{ab}(\sqrt{a} + \sqrt{b}) = ts = 182 \end{aligned}$$

Hence $t^3 = 183 + 3ts = 183 + 546 = 729$ and $t = 9$. It follows that $s = \frac{182}{9}$.
Now,

$$\begin{aligned} a + b &= (\sqrt{a} + \sqrt{b})^2 - 2\sqrt{a}\sqrt{b} \\ &= t^2 - 2s = 81 - \frac{364}{9} = \frac{365}{9} \end{aligned}$$

and $\frac{9}{5}(a+b) = \frac{9}{5} \times \frac{365}{9} = 73$.

3. A contractor has two teams of workers: team A and team B. Team A can complete a job in 12 days and team B can do the same job in 36 days. Team A starts working on the job and team B joins team A after four days. The team A withdraws after two more days. For how many more days should team B work to complete the job?

   Solution In the first four days, team A completes $4 \times \frac{1}{12} = \frac{1}{3}$ of the work. In the next two days, teams A and B together complete

$$2 \times \left(\frac{1}{12} + \frac{1}{36}\right) = \frac{2}{9}$$

   part of the work. Thus the total work completed so far is $\frac{1}{3} + \frac{2}{9} = \frac{5}{9}$. The remaining $\frac{4}{9}$ part of the work can be completed by team B in $\frac{4/9}{1/36} = 16$ days.

4. Let a, b be integers such that all the roots of the equation $(x^2 + ax + 20)(x^2 + 17x + b) = 0$ are negative integers. What is the smallest possible value of $a + b$?

   Solution If $-p, -q$ are the roots of the first quadratic and $-r, -s$, those of the second, then $pq = 20, r + s = 17$. $p+q = a$ is minimum when $p = 4, q = 5$ and $rs = b$ is minimum when $r = 1, s = 16$. Thus the minimum value of $a + b$ is 25.

5. Let u, v, w be real numbers in geometric progression such that $u > v > w$. Suppose $u^{40} = v^n = w^{60}$. Find the value of n.

   Solution Let the progression be $u = \frac{a}{r}, v = a, w = ar$. Given $u^{40} = w^{60}$. This gives $\left(\frac{a}{r}\right)^{40} = (ar)^{60}$. Thus $a =$

r^{-5} and the progression can be written as r^{-6}, r^{-5}, r^{-4}. Thus $v^n = r^{-5n} = u^{40} = r^{-240}$ and $n = 48$.

6. Let the sum $\displaystyle\sum_{n=1}^{9} \frac{1}{n(n+1)(n+2)}$ written in its lowest terms be $\frac{p}{q}$. Find the value of $q - p$.

Solution

$$\begin{aligned}\sum_{n=1}^{9} \frac{1}{n(n+1)(n+2)} &= \sum_{n=1}^{9} \left(\frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+1)}\right) \\ &= \frac{1}{2}\sum_{n=1}^{9} \left(\left(\frac{1}{n} - \frac{1}{n+1}\right) \right. \\ &\qquad \left. - \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right) \\ &= \frac{1}{2}\left(\left(1 - \frac{1}{10}\right) - \left(\frac{1}{2} - \frac{1}{11}\right)\right) \\ &= \frac{27}{110}\end{aligned}$$

Hence $q - p = 83$.

7. Find the number of positive integers n, such that $\sqrt{n} + \sqrt{n+1} < 11$.

Solution Squaring $\sqrt{n} + \sqrt{n+1} < 11$, we get

$$2n + 1 + 2\sqrt{n(n+1)} < 121$$

and hence $\sqrt{n(n+1)} < 60 - n$. Squaring again, we get

$$n(n+1) < 3600 - 120n + n^2 \Rightarrow 121n < 3600$$

Thus $n < \dfrac{3600}{121} = 29.75$ and $n \leq 29$. Thus the number of positive integers satisfying the given inequality is 29. Alternately, from $\sqrt{n(n+1)} < 60 - n$, we deduce

$$n < \sqrt{n(n+1)} < 60 - n \Rightarrow n < 30$$

Since $\sqrt{k(k+1)} < \sqrt{(k+1)(k+2)}$ for all k, it is enough to check that $\sqrt{29} + \sqrt{30} < 11$ to conclude that the maximum n that satisfies the given inequality is 29 and hence the number of integers is 29.

8. A pen costs Rs 11 and a notebook costs Rs 13. Find the number of ways in which a person can spend exactly Rs 1000 to buy pens and notebooks.
Solution We need to find the number of pairs of non negative integers (m, n) such that $11m + 13n = 1000$. Consider the arithmetic progression $1000 - 13n$ for $n = 0, 1, \ldots, 76$. We can find m for each of the term of this sequence that is a multiple of 11. The first few terms are $1000, 987, 974, 961, 948, \ldots$. The remainders when we divide by 11 are

$$10, 8, 6, 4, 2, 0, 9, 7, 5, 3, 1, 10, 8, 6, 4, 2, 0, 9, \ldots$$

Thus the terms $6 + 11k$th terms are multiples of 11. Hence there are $\left\lfloor \frac{77}{11} \right\rfloor = 7$ terms that are multiples of 11. Hence the number of ways is 7.

9. There are five cities A, B, C, D, E on a certain island. Each city is connected to every other city by road. In how many ways can a person starting from city A come back to A after visiting some cities without visiting a city more than once and without taking the same road more than once? (The order in which he visits the cities also matters: for example, the routes $A \to B \to C \to A$ and $A \to C \to B \to A$ are different.)
Solution Note that at least two cities other that A must be visited. For, if only one city, say B is visited before returning to A, then the route must be $A \to B \to A$. In this case, the road $A \to B$ is traversed twice.

If all the cities are visited, then the route is of the form $A \to C_1 \to C_2 \to C_3 \to C_4 \to A$. The cities C_1, C_2, C_3, C_4 can be any permutation of the four cities B, C, D, E and hence there are 24 possibilities. Also, if three of the cities are visited, then the route is of the form $A \to C_1 \to C_2 \to C_3 \to A$. C_1 has 4 possibilities, C_2 has 3 and C_3 has 2. Thus there are 24 routes. If two cities are visited, there are $4 \times 3 = 12$ possibilities. Hence there $24 + 24 + 12 = 60$ ways.

10. There are eight rooms on the first floor of a hotel, with four rooms on each side of the corridor, symmetrically situated (that is, each room is exactly opposite to one other room). Four guests have to be accommodated in four of the eight rooms (that is, one in each) such that no two guests are in adjacent rooms or in opposite rooms. In how many ways can the guests be accommodated?
Solution Let the rooms be R_1, R_2, R_3, R_4 and L_1, L_2, L_3, L_4. Note that in each side there will be exactly two guests (if there are three or more, then by Pigeonhole principle, two of them will be in adjacent rooms). The possible choices of non adjacent rooms on the right side is are $(R_1, R_3), (R_2, R_4), (R_1, R_4)$. But if we place the guests in (R_1, R_4), the only possibility for the other side is (L_2, L_3) and these are adjacent. Hence there are only two possibilities for the right side and for each of these possibilities, there is a unique choice for the other side. Since four guests can be accommodated in 4 rooms in $4! = 24$ ways, the number of ways is $2 \times 24 = 48$.

11. Let $f(x) = \sin \frac{x}{3} + \cos \frac{3x}{10}$. Find the least natural number n such that $f(n\pi + x) = f(x)$ for all x.

Solution Clearly, we need, for all x,

$$\sin\frac{x}{3} = \sin\frac{n\pi + x}{3} \tag{1}$$

$$\cos\frac{3x}{10} = \cos\frac{3(n\pi + x)}{10} \tag{2}$$

Equation (1) is satisfied when $n = 6k$ and (2) is satisfied when $n = 20k$, for any integer k. Hence both are satisfied when $n = 60$.

12. In a class, the total numbers of boys and girls are in the ratio $4 : 3$. On one day it was found that 8 boys and 14 girls were absent from the class, and that the number of boys was the square of the number of girls. What is the total number of students in the class?

Solution Let the number of boys and girls in the class be $4n, 3n$ respectively, where n is a positive integer. Given $4n - 8 = (3n - 14)^2$. Solving, $n = 6$ or $\frac{34}{9}$. Discarding the non integer root, the number of students in the class is $7n = 42$.

13. In a rectangle $ABCD$, E is the midpoint of AB; F is a point on AC such that BF is perpendicular to AC; and FE perpendicular to BD. Suppose $BC = 8\sqrt{3}$. Find AB.

Solution Triangle AFB is right angled and E is the

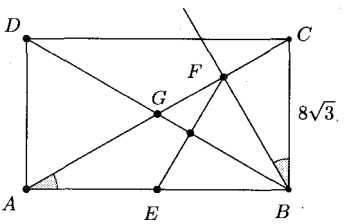

Figure 27

mid point of the hypotenuse. Hence $AE = FE = EB$ and $\angle AFE = \angle FAE$. Also,

$$\angle DBE = \angle FAE = \angle AFE = 90^\circ - \angle FGB = \angle GBF$$

Thus BF, BD trisect $\angle CBA$ and $\angle CAB = 30^\circ$. Thus $AB = BC \cot 30^\circ = 24$.

14. Suppose x is a positive real number such that $\{x\}, [x]$ and x are in a geometric progression. Find the least positive integer n such that $x^n > 100$. (Here $[x]$ denotes the integer part of x and $\{x\} = x - [x]$.)

Solution Let $x = m + f$, where m is an integer and $0 \leq f < 1$. Then $[x] = m, \{x\} = f$. We need $m^2 = f(m + f)$. This gives $f^2 + mf - m^2 = 0$. Hence $f = \dfrac{-m \pm m\sqrt{5}}{2}$. Since $0 \leq f < 1$, it follows that $f = m\left(\frac{\sqrt{5}-1}{2}\right)$. It follows that $m = 1$ (using $f < 1$). Thus

$$x = 1 + \left(\frac{\sqrt{5}-1}{2}\right) = \frac{\sqrt{5}+1}{2}$$

One easily sees that $x^4 < 7$ and $x^5 > 10$. Thus $x^9 < 49x < 98$ and $x^{10} > 100$. Thus the least integer n such that $x^n > 100$ is 10.

15. Integers $1, 2, 3, \ldots, n$ where $n > 2$, are written on a board. Two numbers m, k such that $1 < m < n$ and $1 < k < n$ are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers?

Solution When two numbers $1 < m, k < n$ are removed from $1, 2, \ldots, n$, the minimum sum occurs when the removed numbers are $n - 2, n - 1$ and maximum sum occurs when the removed numbers are $2, 3$. Thus we

have

$$\frac{\frac{n(n+1)}{2} - (2n-3)}{n-2} \leq 17 \leq \frac{\frac{n(n+1)}{2} - 5}{n-2}$$

$$\Rightarrow \frac{n^2 - 3n + 6}{2(n-2)} \leq 17 \leq \frac{n^2 + n - 10}{2(n-2)}$$

$$\Rightarrow 31 \leq n \leq 34$$

Now,

$$\frac{\frac{n(n+1)}{2} - (m+k)}{n-2} = 17$$

$$\Rightarrow m + k = \frac{n(n+1)}{2} - 17(n-2)$$

The maximum $m + k$ happens when $n = 34$ and the maximum value is 51.

16. Five distinct 2-digit numbers are in a geometric progression. Find the middle term.

Solution Let the numbers be a, ar, ar^2, ar^3, ar^4. Given that $a \geq 10$ and $ar^4 < 100$. Thus $10r^4 < 100$ and $r^4 < 10$. Note that r is rational. Let $r = p/q$, where p, q have no common divisors $\neq 1$. q^4 must divide a. Thus $q < 4$. If $q = 1$, from $p^4 < 10$, we get $p = 1$, and $r = 1$, a contradiction since the numbers have to be distinct. If $q = 3$, then $a = 81$. Note that $p = 1$ and the progression becomes $81, 27, 9, 3, 1$ – not all have two digits. Thus $q = 2$. This gives a is a multiple of 16. Since $p^4 < 10q^4$, we have $p \leq 3$. It follows that $p = 3$ and the progression is $16, 24, 36, 54, 81$ and the middle term is 36.

17. Suppose the altitudes of a triangle are 10, 12 and 15. What is its semi-perimeter?

Solution If the sides are a, b, c respectively, then $10a =$

$12b = 15c = 2A$, where A is the area of the triangle. Hence we have

$$\frac{10a}{60} = \frac{12b}{60} = \frac{15c}{60}$$
$$\Rightarrow \frac{a}{6} = \frac{b}{5} = \frac{c}{4} = \frac{A}{30} = k \text{ (say)}$$

Hence $a = 6k, b = 5k, c = 4k$ and the semi perimeter $s = \frac{15}{2}k$.

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \frac{15\sqrt{7}}{4}k^2 = 30k$$

This gives $k = \frac{8}{\sqrt{7}}$ and the semi perimeter is $\frac{15}{2}k = \frac{60}{\sqrt{7}}$. The result is not an integer. This question was discounted.

18. If the real numbers x, y, z are such that $x^2+4y^2+16z^2 = 48$ and $xy + 4yz + 2zx = 24$, what is the value of $x^2 + y^2 + z^2$?

Solution

$$2(x^2 + 4y^2 + 16z^2) - 4(xy + 4yz + 2zx) = 0$$
$$\Rightarrow (x - 2y)^2 + (2y - 4z)^2 + (4z - x)^2 = 0$$

and hence $x = 2y = 4z$. This gives $48z^2 = 48$ and $z^2 = 1$. Thus $x = \pm 4, y = \pm 2, z = \pm 1$ and $x^2 + y^2 + z^2 = 21$.

19. Suppose $1, 2, 3$ are the roots of the equation $x^4 + ax^2 + bx = c$. Find the value of c.

Solution Since the coefficient of x^3 is zero, sum of the roots of the equation is 0. Thus the fourth root is -6. Also, product of the roots is $-c$ and hence $-c = -36$ and $c = 36$.

20. What is the number of triples (a, b, c) of positive integers such that $(i)\, a < b < c < 10$ and $(ii)\, a, b, c, 10$ form the sides of a quadrilateral?

    Solution Since $c < 10$, (a, b, c) can be any three distinct numbers from $1, 2, \ldots, 9$ and such that $a + b + c > 10$. The number of ways of choosing 3 distinct numbers from $1, 2, \ldots, 9$ is $\binom{9}{3} = 84$. From this, we need to eliminate those cases for which $a + b + c \leq 10$. Thus we need to eliminate the cases for which $a + b + c = 6, 7, 8, 9, 10$. We need to eliminate the cases $(1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 4), (1, 2, 6), (1, 3, 5), (2, 3, 4), (1, 2, 7), (1, 4, 5), (1, 3, 6), (1, 3, 6)$ and $(2, 3, 5)$. Hence there are 11 cases to eliminate and the required number is $84 - 11 = 73$.

21. Find the number of ordered triples (a, b, c) of positive integers such that $abc = 108$.

    Solution Since $108 = 3^3 \times 2^2$, each of a, b, c is of the form $3^i \times 2^j$, where $0 \leq i \leq 3$ and $0 \leq j \leq 2$. The number of non negative integer solutions to $i_1 + i_2 + i_3 = 3$ is $\binom{5}{2}$ and the number of non negative integer solutions to $j_1 + j_2 + j_3 = 2$ is $\binom{4}{2}$. Thus the number of ordered triples is $\binom{5}{2} \times \binom{4}{2} = 60$.

22. Suppose in the plane 10 pairwise nonparallel lines intersect one another. What is the maximum possible number of polygons (with finite areas) that can be formed?

    Solutions The maximum number of finite regions will occur when any k lines are such that their intersections

form a k sided polygon. Thus the largest number is

$$\begin{aligned}&\binom{10}{3}+\binom{10}{4}+\cdots+\binom{10}{10}\\&=2^{10}-\binom{10}{0}-\binom{10}{1}-\binom{10}{2}\\&=968\end{aligned}$$

Since the answer is not a two digit integer, the question was discounted.

23. Suppose an integer x, a natural number n and a prime number p satisfy the equation $7x^2-44x+12=p^n$. Find the largest value of p.

Solution We have $7x^2-44x+12=(7x-2)(x-6)$.

Case 1 $n=1$

Here, either $7x-2=1$ or $x-6=1$. Clearly $7x-2=1$ yields $x=\frac{3}{7}$, not possible, since x must be an integer. Hence $x-6=1$ and $x=7$. Thus gives $p=47$.

Case 2 $n>1$.

Since one of $7x-2$ or $x-6$ being 1 has already been considered as part of **Case 1**, let us look at the case p divides both $7x-2$ and $x-6$. Then p divides $(7x-2)-7(x-6)=40$ and hence $p=5$ or 2. Thus the maximum p is 47.

24. Let P be an interior point of a triangle ABC whose side lengths are $26, 65, 78$. The line through P parallel to BC meets AB in K and AC in L. The line through P parallel to CA meets BC in M and BA in N. The line through P parallel to AB meets CA in S and CB in T. If KL, MN, ST are of equal lengths, find this common length.

Solution Let $MN=KL=ST=a$. Triangles ALK

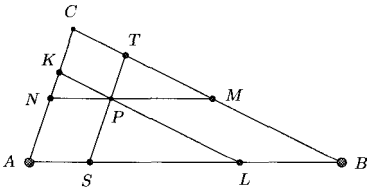

Figure 28

and ACB are similar. Hence $AL = 6a/5, AK = 2a/5$. Similarly, from the similar triangles CST and CAB, we get $CS = 3a, CT = 5a/2$ and from similar triangles BNM and BAC, we get $BN = a/3, BM = 5a/6$. Now, $SL = AL + SC - AC = \dfrac{21a}{5} - 78$. Thus $AS = AL - SL = 78 - 3a$ and $CL = SC - SL = 78 - \dfrac{6a}{5}$. Also, $AS + CL = NP + PM$ which gives

$$(78 - 3a) + \left(78 - \frac{6a}{5}\right) = a \Rightarrow a = 30$$

This is not possible, since the common length must be less than the shortest side, 28, of the triangle ABC. This question was discounted.

25. Let $ABCD$ be a rectangle and let E and F be points on CD and BC respectively such that area $(ADE) = 16$, area $(CEF) = 9$ and area $(ABF) = 25$. What is the area of triangle AEF?

Solution With A as the origin of vectors, let the position vectors of B and D be respectively $\mathbf{b}$ and $\mathbf{d}$. Let $BF : FC = \mu : 1$ and $DE : EC = \lambda : 1$. Then the position vector of E is $\mathbf{d} + \dfrac{\lambda}{\lambda + 1}\mathbf{b}$ and that of F is $\mathbf{b} + \dfrac{\mu}{\mu + 1}\mathbf{d}$.

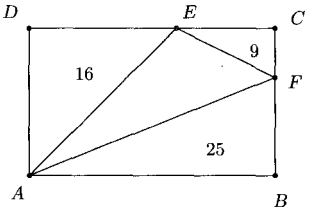

Figure 29

We have

$$2\text{Area of } ADE = \left|\left(\mathbf{d} + \frac{\lambda}{\lambda+1}\mathbf{b}\right) \times \mathbf{d}\right| = \frac{\lambda}{\lambda+1}|\mathbf{b} \times \mathbf{d}|$$

$$2\text{Area of } AFB = \left|\mathbf{b} \times \left(\mathbf{b} + \frac{\mu}{\mu+1}\mathbf{d}\right)\right| = \frac{\mu}{\mu+1}|\mathbf{b} \times \mathbf{d}|$$

$$2\text{Area of } CEF = \left|\left(\frac{1}{\lambda+1}\mathbf{b}\right) \times \left(\frac{1}{\mu+1}\mathbf{d}\right)\right|$$

$$= \frac{1}{(\lambda+1)(\mu+1)}|\mathbf{b} \times \mathbf{d}|$$

Hence

$$\frac{\lambda(\mu+1)}{\mu(\lambda+1)} = \frac{16}{25}$$

$$\lambda(\mu+1) = \frac{16}{9}$$

Solving for λ, μ, we get $\lambda = \frac{2}{3}, \mu = \frac{5}{3}$. This gives

$$\text{Area of } ABCD = |\mathbf{b} \times \mathbf{d}| = 80$$

Hence area of $AEF = 80 - 16 - 9 - 25 = 30$.

26. Let AB and CD be two parallel chords in a circle with radius 5 such that the center O lies between these chords. Suppose $AB = 6, CD = 8$. Suppose further that the area

of the part of the circle lying between the chords AB and CD is $\frac{m\pi + n}{k}$ where m, n, k are positive integers with $\gcd(m, n, k) = 1$. What is the value of $m + n + k$?

Solution Let E be the foot of the perpendicular from A

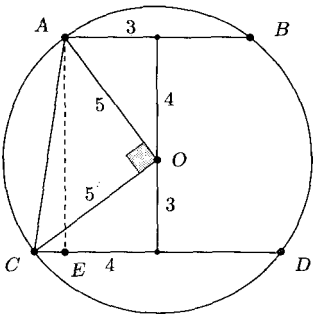

Figure 30

on CD. Then AE equals distance between the parallel chords AB and CD and hence equals 7. Also $CE = 1$ and thus $AC = \sqrt{7^2 + 1^2} = 5\sqrt{2}$. Thus $\angle AOC = \frac{\pi}{2}$. Let $\angle AOB = \alpha$ and $\angle COD = \beta$. Then $\alpha + \beta = 2\pi - 2\angle AOC = \pi$.

The area between the chords is given by

$$2 \times \left(\frac{1}{2} \times 25 \times \frac{\pi}{2}\right) + 24 = \frac{25\pi + 48}{2}$$

and $m + n + k = 75$.

27. Let Ω_1 be a circle with center O and let AB be a diameter of Ω_1. Let P be a point on the segment OB different from O. Suppose another circle is Ω_2 with center P lies in the interior of Ω_1. Tangents are drawn from A and B to the circle Ω_2 intersecting Ω_1 again at A_1 and B_1 respectively such that A_1 and B_1 are on the opposite sides of AB. Given that $A_1B = 5, AB_1 = 15$

and $OP = 10$, find the radius of is Ω_1.

Solution Let R_1, R_2 be the radii of the circles Ω_1, Ω_2

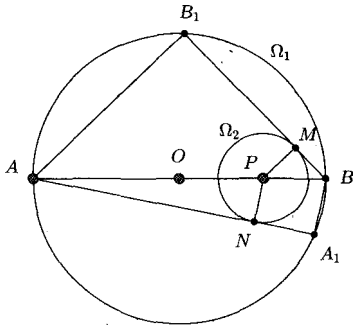

Figure 31

respectively. Let M, N be the feet of the perpendiculars on the tangents from P. Since triangles APN and ABA_1 are similar, we have $\frac{R_2}{5} = \frac{R_1 + 10}{2R_1}$. Similarly, from the similar triangles BPM and BAB_1, we get $\frac{R_2}{15} = \frac{R_1 - 10}{2R_1}$. These give $\frac{R_1 - 10}{R_1 + 10} = \frac{5}{15}$ and hence $R_1 = 20$.

28. Let p, q be prime numbers such that $n^{3pq} - n$ is a multiple of $3pq$ for all positive integers n. Find the least possible value of $p + q$.

Solution We will use the following fact from Number Theory:

If P is any prime and $a \neq 1$ is prime to P satisfies $a^d = 1 \pmod{P}$ for some positive integer d, then $P - 1$ divides d.

Letting $n = 2$, we get $2^{3pq} - 2 = 2(2^{3pq-1} - 1)$ is a multiple of $3pq$. Thus $2^{3pq-1} = 1 \pmod{3}$ and it follows that 2 divides $3pq - 1$. Hence p, q are both odd primes. Similarly, letting $n = p - 1$ and $q - 1$, we obtain

respectively, $p-1$ and $q-1$ divide $3pq-1$. Again, since $3pq-1=3(p-1)q+3q-1=3(q-1)p+3p-1$, it follows that $p-1$ divides $3q-1$ and $q-1$ divides $3p-1$.

Without loss of generality, we can assume $p<q$. We claim $p>3$. If not, $3^{9q}-3$ must be a multiple of $9q$, a contradiction.

Looking for primes p,q such that $p-1$ divides $3q-1$ and $q-1$ divides $3p-1$, we obtain the least possible values for p,q as $p=11$ and $q=17$. Thus the least possible value of $p+q$ is 28.

29. For each positive integer n, consider the highest common factor h_n of the two numbers $n!+1$ and $(n+1)!$. For $n<100$, find the largest value of h_n.

Solution By Wilson's theorem, a positive integer N is a prime if and only if $(N-1)!+1$ is a multiple of N. Hence if $n+1$ is a prime, $n+1$ divides $n!+1$ and the highest common factor of $n!+1$ and $(n+1)!$ is $n+1$. Since $(n+1)(n!+1)-(n+1)!=n+1$, any common divisor of $n!+1$ and $(n+1)!$ must be a divisor of $n+1$. Thus the maximum highest common factor will occur when $n+1$ is a prime. Since $n<100$, the highest prime is 97 and when $n=96$, we have $h_n=97$.

30. Consider the areas of the four triangles obtained by drawing the diagonals AC and BD of a trapezium $ABCD$. The product of these areas, taken two at time, are computed. If among the six products so obtained, two products are 1296 and 576, determine the square root of the maximum possible area of the trapezium to the nearest integer.

Solution Let the diagonals AC and BD meet at O.

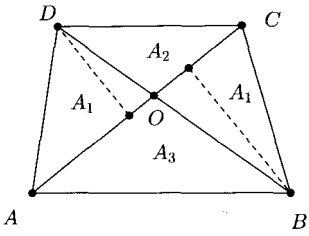

Figure 32

Since the triangles ABC and ABD are on the same base and between the parallels AB and CD, their areas are equal. Also,

$$\begin{aligned} \text{Area of } BOC &= \text{ Area of } ABC - \text{ Area of } AOB \\ &= \text{ Area of } ABD - \text{ Area of } AOB \\ &= \text{ Area of } AOD \end{aligned}$$

Thus the four areas are A_1, A_1, A_2, A_3 as shown in Figure 32. The six products of the areas taken two at a time are $A_1^2, A_1A_2, A_1A_3, A_1A_2, A_2A_3, A_1A_3$. However, note that $\dfrac{A_1}{A_2} = \dfrac{A_3}{A_1}$ and hence $A_2A_3 = A_1^2$. Thus among the products, only three are distinct: A_1^2, A_1A_2, A_1A_3.

Case 1 $A_1^2 = 1296$, $A_1A_2 = 576$

In this case, $2A_1 + A_2 + A_3 = 169$

Case 2 $A_1^2 = 576$, $A_1A_2 = 1296$

In this case, $2A_1 + A_2 + A_3 = 112.66 < 169$

Case 3 $A_1^2 = 1296$, $A_2A_3 = 576$

In this case, $2A_1 + A_2 + A_3 = 25\sqrt{24} < 169$

Hence the square root of the maximum possible area of the trapezium is 13.

RMO 2017

1. Let AOB be a given angle less than $180°$ and let P be an interior point of the angular region determined by $\angle AOB$. Show, with proof, how to construct, using only ruler and compasses, a line segment CD passing through P such that C lies on the ray OA and D lies on the ray OB, and $CP : PD = 1 : 2$.

   Solution Draw a line parallel to OA through P. Let

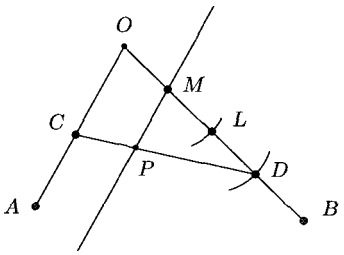

Figure 33

it intersect OB in M. Using compasses, draw an arc of a circle with centre M and radius MO to cut OB in L, $L \neq O$. Again with L as centre and with the same radius OM draw one more arc of a circle to cut OB in D, $D \neq M$. Join DP and extend it to meet OA in C. Then CD is the required line segment such that $CP : PD = 1 : 2$. This follows from similar triangles OCD and MPD.

2. Show that the equation

$$a^3+(a+1)^3+(a+2)^3+(a+3)^3 \\ +(a+4)^3+(a+5)^3+(a+6)^3 = b^4+(b+1)^4$$

has no solutions in integers a, b.

Solution We use divisibility argument by 7. Observe that the remainders of seven consecutive cubes modulo 7 are $0, 1, 1, 6, 1, 6, 6$ in some (cyclic) order. Hence the sum of seven consecutive cubes is 0 modulo 7. On the other hand the remainders of two consecutive fourth powers modulo 7 is one of the sets $\{0,1\}, \{1,2\}, \{2,4\}, \{4,4\}$. Hence the sum of two fourth powers is never divisible by 7. It follows that the given equation has no solution in integers.

3. Let $P(x) = x^2 + \frac{1}{2}x + b$ and $Q(x) = x^2 + cx + d$ be two polynomials with real coefficients such that $P(x)Q(x) = Q(P(x))$ for all real x. Find all the real roots of $P(Q(x)) = 0$.

Solution Since $P(x)$ divides

$$Q(P(x)) = (P(x))^2 + cP(x) + d$$

it follows that $P(x)$ divides d and hence $d = 0$. Thus $P(x)Q(x) = P(x)^2 + cP(x)$ and hence $Q(x) = P(x) + c$. This gives $c = Q(x) - P(x) = \left(c - \frac{1}{2}\right)x - b$ and thus $c = \frac{1}{2}$ and $b = -c = -\frac{1}{2}$. Consequently, $P(x) = x^2 + \frac{1}{2}x - \frac{1}{2}$ and $Q(x) = x^2 + \frac{1}{2}x$ and

$$\begin{aligned} P(Q(x)) &= \left(x^2 + \frac{1}{2}x\right)^2 + \frac{1}{2}\left(x^2 + \frac{1}{2}x\right) - \frac{1}{2} \\ &= \frac{1}{4}(4x^4 + 4x^3 + 3x^2 + x - 2) \end{aligned}$$

Clearly $P(Q(-1)) = 0$ and any other rational root will be $\pm\frac{1}{4}, \pm\frac{1}{2}$. Of these, $\frac{1}{2}$ is a root. The remaining factor is $h(x) = x^2 + \frac{1}{2}x + 1$ and this has no real roots.

4. Consider n^2 unit squares in the xy plane centered at point (i, j) with integer coordinates, $1 \leq i \leq n$ and

$1 \leq j \leq n$. It is required to color each unit square in such a way that whenever $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, the three squares with centers at $(i,k),(j,k),(j,l)$ have distinct colors. What is the least possible number of colors needed?

Solution We first show that at least $2n-1$ colors are needed. Observe that squares with centers $(i,1)$ must all have different colors for $1 \leq i \leq n$. Let us call them $C_1, C_2, \ldots, C_n$. Besides, the squares with centers (n,j), for $2 \leq j \leq n$ must have different colors and these must be different from $C_1, C_2, \ldots, C_n$. Thus we need at least $n+(n-1) = 2n-1$ colors. Figure 34 shows that $2n-1$ colors will suffice.

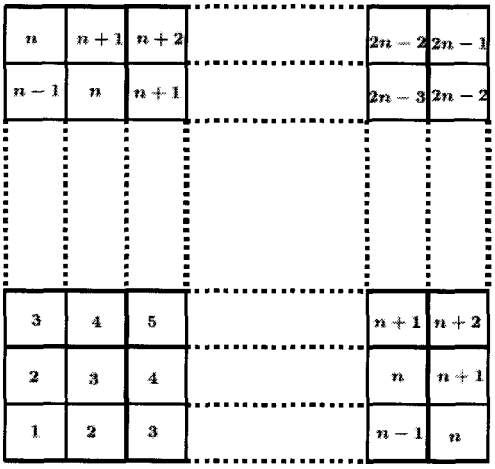

Figure 34

5. Let Ω be a circle with a chord AB which is not a diameter. Let Γ_1 be a circle on one side of AB such that it is tangent to AB at C and internally tangent to Ω at D. Likewise, let Γ_2 be a circle on the other side of

AB such that it is tangent to AB at E and internally tangent to Ω at F. Suppose the line DC intersects Ω at $X \neq D$ and the line FE intersects Ω at $Y \neq F$. Prove that XY is a diameter of Ω.

Solution Let the centers of the circles Γ_1 and Γ_2 be

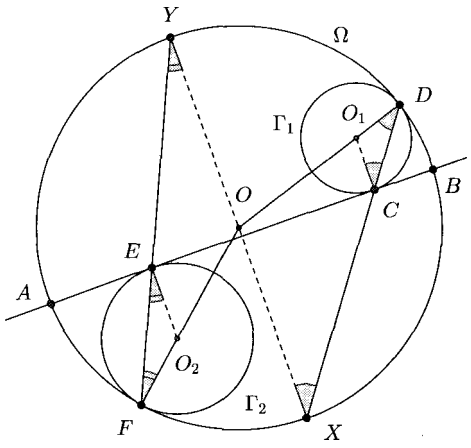

Figure 35

O_1 and O_2 respectively. Since Γ_1 touches AB at C, $O_1C \perp AB$. Similarly, $O_2E \perp AB$. Thus $O_1C \parallel O_2E$. D lies on the line joining the centers OO_1. In triangle O_1CD, $O_1D = O_1C$ and hence $\angle O_1DC = \angle O_1CD$. In triangle ODX, $OD = OX$ and hence $\angle ODX = \angle OXD$. Thus

$$\angle OXD = \angle ODX = \angle O_1DC = \angle O_1CD$$

and thus OX is parallel to O_1C.

Applying a similar argument to Γ_2, we get $O_2E \parallel OY$. Since $O_1C \parallel O_2E$, we obtain $OX \parallel OY$. Since O is the center of Ω, it follows that XY is a diameter of Ω.

6. Let x, y, z be real numbers each greater than 1. Prove that

$$\frac{x+1}{y+1}+\frac{y+1}{z+1}+\frac{z+1}{x+1} \leq \frac{x-1}{y-1}+\frac{y-1}{z-1}+\frac{z-1}{x-1}$$

Solution The inequality is equivalent to

$$\left(\frac{x-1}{y-1}-\frac{x+1}{y+1}\right)+\left(\frac{y-1}{z-1}-\frac{y+1}{z+1}\right) + \left(\frac{z-1}{x-1}-\frac{z+1}{x+1}\right) \geq 0$$

and hence we need to prove

$$\frac{x-y}{y^2-1}+\frac{y-z}{z^2-1}+\frac{z-x}{x^2-1} \geq 0$$

or equivalently,

$$\frac{x}{y^2-1}+\frac{y}{z^2-1}+\frac{z}{x^2-1} \geq \frac{y}{y^2-1}+\frac{z}{z^2-1}+\frac{x}{x^2-1}$$

By symmetry, we can assume that $x \geq y \geq z$. Then

$$\frac{1}{x^2-1} \leq \frac{1}{y^2-1} \leq \frac{1}{z^2-1}$$

Hence by rearrangement inequality, the result follows.

Note If $a_1 \geq a_2 \geq a_3$ and $b_1 \leq b_2 \leq b_3$ and c_1, c_2, c_3 is any permutation of b_1, b_2, b_3, rearrangement inequality states that the following inequality holds:

$$a_1b_1 + a_2b_2 + a_3b_3 \leq a_1c_1 + a_2c_2 + a_3c_3$$

Here, we apply the rearrangement inequality with

$$(a_1, a_2, a_3) = (x, y, z)$$
$$(b_1, b_2, b_3) = \left(\frac{1}{x^2-1}, \frac{1}{y^2-1}, \frac{1}{z^2-1}\right)$$
$$(c_1, c_2, c_3) = \left(\frac{1}{y^2-1}, \frac{1}{z^2-1}, \frac{1}{x^2-1}\right)$$

INMO 2018

1. Let ABC be a non-equilateral triangle with integer sides. Let D and E be respectively the mid-points BC and CA; let G be the centroid of triangle ABC. Suppose D, C, E, G are concyclic. Find the least possible perimeter of triangle ABC.

Solution Let the lengths of the sides of the triangle

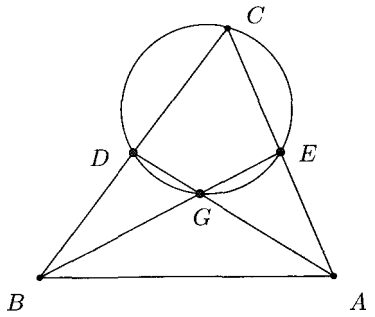

Figure 36

be a, b, c. Since D, C, E, G are concylcic, $BD \cdot BC = BG \cdot BE$ and hence $\frac{a^2}{2} = \frac{2}{3}BE^2$. Since $BE^2 = \frac{1}{4}(2a^2 + 2c^2 - b^2)$, it follows that

$$\frac{1}{4}(2a^2 + 2c^2 - b^2) = \frac{3a^2}{4} \Longrightarrow a^2 + b^2 = 2c^2$$

Thus a, b have the same parity and

$$\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2 = c^2$$

This implies that $\left(\frac{a+b}{2}, \frac{a-b}{2}, c\right)$ is a Pythagorean triple. To search for the triangle with the smallest

perimeter, it is sufficient to look at the primitive triples. For the triple $(3,4,5)$, we get $a = 7, b = 1, c = 5$ but there is no triangle with these side lengths. The next triple $(13,12,5)$ gives the triangle with side lengths $17, 7, 13$ and perimeter 37. Noting that $2c < a+b+c$ and we already have a triangle with perimeter 37, to find the smallest perimeter it is sufficient to check for Pythagorean triples with $c \leq 18$. But the next primitive triple is $(7,24,25)$ and hence the minimum perimeter is 37.

2. For any natural number n, consider a rectangular board made up of n unit squares. This is covered by three types of tiles, a 1×1 red tile, 1×1 green tile and a 2×1 blue domino. Let t_n denote the number of ways of covering $1 \times n$ rectangular board by these three types of tiles. Prove that t_n divides t_{2n+1}.

Solution 1 Let r_n, g_n, b_n respectively be the number of $1 \times n$ tiles that end with a red, green and blue tiles. Clearly, $t_n = r_n + g_n + b_n$. To get a $1 \times (n+1)$ tile ending in a red tile, we can append a 1×1 red tile to any of the above three. Hence $r_{n+1} = r_n + g_n + b_n$. Similarly, $g_{n+1} = r_n + g_n + b_n$. To get b_{n+1}, we need to append a blue tile to a $1 \times (n-1)$ tile. Thus $b_{n+1} = r_{n-1} + g_{n-1} + b_{n-1}$. Thus

$$\begin{aligned} t_{n+1} &= r_{n+1} + g_{n+1} + b_{n+1} \\ &= (r_n + g_n + b_n) + (r_n + g_n + b_n) \\ &\qquad + (r_{n-1} + g_{n-1} + b_{n-1}) \\ &= 2t_n + t_{n-1} \end{aligned}$$

Thus we have the recurrence relation $t_{n+1} - 2t_n - t_{n-1} = 0$ whose characteristic equation is $\lambda^2 - 2\lambda - 1 = 0$. This

has characteristic roots $1 \pm \sqrt{2}$. Thus

$$t_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n = A\alpha^n + B\beta^n$$

where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$. Since $t_1 = 2$ and $t_2 = 5$, we get $A = \dfrac{\alpha}{2\sqrt{2}}$ and $B = -\dfrac{\beta}{2\sqrt{2}}$. Hence

$$t_n = \frac{\alpha^{n+1} - \beta^{n+1}}{2\sqrt{2}}$$

Now,

$$\begin{aligned} t_{2n+1} &= \frac{\alpha^{2n+2} - \beta^{2n+2}}{2\sqrt{2}} \\ &= \frac{(\alpha^{n+1} - \beta^{n+1})(\alpha^{n+1} + \beta^{n+1})}{2\sqrt{2}} \\ &= \left(\frac{\alpha^{n+1} - \beta^{n+1}}{2\sqrt{2}}\right)(\alpha^{n+1} + \beta^{n+1}) \\ &= t_n(\alpha^{n+1} + \beta^{n+1}) \end{aligned}$$

Note that

$$\begin{aligned} \alpha^{n+1} + \beta^{n+1} &= (1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1} \\ &= 2\left(1 + \binom{n+1}{2} \cdot 2 + \binom{n+1}{4} \cdot 2^2 + \cdots\right) \end{aligned}$$

is an integer and hence t_{2n+1} is divisible by t_n.

Solution 2 Consider a $1 \times (2n+1)$ board and imagine the board to be placed horizontally. Let us label the squares of the board as

$$C_{-n}, C_{-(n-1)}, \ldots, C_{-1}, C_0, C_1, \ldots, C_n$$

from left to right. The 1×1 tiles will be referred to as tiles, and the blue 1×2 tile will be referred to as a domino.

Let us consider the different ways in which the center square C_0 can be covered. There are four distinct ways in which this can be done:

(a) There is a blue domino covering the squares C_{-1}, C_0. In this case, there is a $1 \times (n-1)$ board remaining on the left of this domino which can be covered in t_{n-1} ways, and there is a $1 \times n$ board remaining on the right of the domino which can be covered in t_n ways.

(b) There is a blue domino covering the squares C_0, C_1. In this case, there is a $1 \times n$ board remaining on the left of this domino which can be covered in t_n ways, and there is a $1 \times (n-1)$ board remaining on the right of the domino which can be covered in t_{n-1} ways.

(c) There is a red tile covering the square C_0. In this case, there is a $1 \times n$ board remaining on both sides of this tile, each of which can be covered in t_n ways.

(d) There is a green tile covering the square C_0. In this case, there is a $1 \times n$ board remaining on both sides of this tile, each of which can be covered in t_n ways.

Putting all the possibilities mentioned above together, we get that $t_{2n+1} = 2t_{n-1}t_n + 2t_n^2 = t_n(2t_{n-1} + 2t_n)$ which implies that t_n divides t_{2n+1}.

Solution 3 We can build a $1 \times (2n+1)$ board in two ways:

(a) Take a $1 \times n$ board and another $1 \times (n+1)$ board and stick them together. This gives $t_n \times t_{n+1}$ possibilities.

(b) Take a $1 \times (n-1)$ board, a blue domino, another $1 \times (n-1)$ board and stick them in order with blue domino in the middle. This gives $t_{n-1} \times t_n$ possibilities.

Thus $t_{2n+1} = t_n t_{n+1} + t_{n-1} t_n = t_n(t_{n-1} + t_{n+1})$ and it follows that t_n divides t_{2n+1}.

3. Let Γ_1 and Γ_2 be two circles with respective centers O_1 and O_2 intersecting in two distinct points A and B such that $\angle O_1 A O_2$ is an obtuse angle. Let the circumcircle of triangle $O_1 A O_2$ intersect Γ_1 and Γ_2 respectively in points C and D. Let the line CB intersect Γ_2 in E; let the line DB intersect Γ_1 in F. Prove that the points C, D, E, F are concyclic.

Solution We will first prove that C, B, O_2, E are

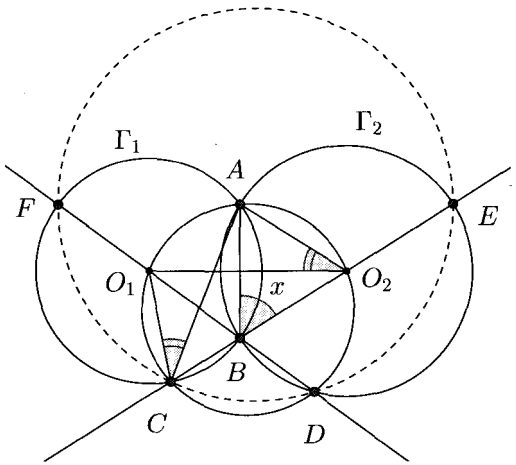

Figure 37

collinear and that this line bisects $\angle ACD$.

Let $\angle ABO_2 = x$. Since $O_2A = O_2B$, $\angle O_2AB = x$ and thus $\angle AO_2B = 180° - 2x$. This gives $\angle AO_2O_1 = 90° - x$. Since A, O_1, C, O_2 are concyclic, we obtain $\angle AO_2O_1 = \angle ACO_1 = 90° - x$. Therefore, $\angle AO_1C = 2x$. From this, we get $\angle AFC = x$ and $\angle ABC = 180° - x$. Thus, $\angle ABC$ and $\angle ABO_2$ are supplementary, implying C, B, O_2, E are collinear. Finally, we note that $O_2A = O_2D$ implies that O_2 is the midpoint of arc AO_2D, and hence CO_2 is the bisector of $\angle ACD$, as required.

Similarly we obtain that D, B, O_1, F are collinear. Hence, BE and BF are diameters of the respective circles. This shows that $\angle BAE = \angle BAF = 90°$ and hence F, A, E are collinear. Finally,

$$\angle ECD = \angle BCD = \angle ACB = \angle AFB = \angle EFD$$

Therefore C, D, E, F are concyclic, as required.

4. Find all polynomials with real coefficients $P(x)$ such that $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

Solution We will show that $P(x) = kx^n$ for some constant k and $n \geq 0$.

Suppose that $\alpha \neq 0$ is a root of $P(x) = 0$. Let β_1, β_2 be the roots of the quadratic $x^2 + x + 1 = \alpha$.

Since $P(x^3 - 1) = P(x^2 + x + 1)Q(x)$, we get $P(\beta_i^3 - 1) = P(\alpha)Q(\beta_i) = 0$ for $i = 1, 2$. Thus both $\beta_1^3 - 1$ and $\beta_2^3 - 1$ are roots of $P(x) = 0$.

Also, $\beta_1 + \beta_2 = -1$ and

$$\beta_1^3 - 1 = (\beta_1 - 1)(\beta_1^2 + \beta_1 + 1) = \alpha(\beta_1 - 1)$$

Similarly, $\beta_2^3 - 1 = \alpha(\beta_2 - 1)$. Thus

$$\begin{aligned} |\beta_1^3 - 1| + |\beta_2^3 - 1| &\geq |(\beta_1^3 - 1) + (\beta_2^3 - 1)| \\ &= |\alpha(\beta_1 + \beta_2 - 2)| \\ &= 3|\alpha| \end{aligned}$$

Thus at least one of $|\beta_1^3 - 1|$ or $|\beta_2^3 - 1|$ must be greater than $\frac{3}{2}|\alpha|$. Let this be α_2. Thus we have found a root α_2 of $P(x) = 0$ such that $|\alpha_2| > |\alpha|$. Repeating this argument, we can find a sequence of roots α_n with

$$|\alpha| < |\alpha_2| < |\alpha_3| < \cdots$$

Since $P(x)$ is a polynomial, it can have only finitely many roots. It follows that $P(x) = 0$ can not have a non zero root. Thus $P(x) = kx^n$ for some constant k and $n \geq 0$.

5. There are $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbors combined, the teacher takes away one apple from that girl and gives one apple each to her neighbors. Prove that this process stops after a finite number of steps. (Assume that the teacher has an abundant supply of apples.)

Solution Let $a_1, a_2, \ldots, a_n$ denote the number of apples with these girls at any given time, all taken in a circular way. Consider two quantities associated with this distribution: $s = a_1 + a_2 + \cdots + a_n$ and $t = a_1^2 + a_2^2 + \cdots + a_n^2$. Using Cauchy-Schwarz inequality, we see that

$$nt = n(a_1^2 + a_2^2 + \cdots + a_n^2) \geq (a_1 + a_2 + \cdots + a_n)^2 = s^2$$

Therefore $t \geq \dfrac{s^2}{n}$ at any stage of the above process. Whenever teacher makes a move, s increases by 1. Suppose the girl with a_j apples has more than the sum of her neighbors. Then the change in t equals

$$\begin{aligned}(a_j - 1)^2 + (a_{j-1} + 1)^2 + (a_{j+1} + 1)^2 - a_j^2 - a_{j-1}^2 - a_{j+1}^2 \\ &= 2(a_{j+1} + a_{j-1} - a_j) + 3 \\ &\leq 2(-1) + 3 = 1\end{aligned}$$

If s_1 and t_1 denote the corresponding sums after one move, we see that

$$s_1 = s + 1 \qquad t_1 \leq t + 1$$

Thus, after k moves, if the corresponding sums are s_k, t_k, then we have

$$t + k \geq t_k \geq \frac{s_k^2}{n} = \frac{(s+k)^2}{n}$$

This leads to a quadratic in k:

$$k^2 + k(2s - n) + (s^2 - nt) \leq 0$$

Since this can not hold for large k, the process must stop after some stage.

6. Let $\mathbb{N}$ denote the set of all natural numbers and let $f : \mathbb{N} \to \mathbb{N}$ be a function such that

   (a) $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$

   (b) $m + n$ divides $f(m) + f(n)$ for all $m, n \in \mathbb{N}$

   Prove that there exists an odd natural number k such that $f(n) = n^k$ for all $n \in \mathbb{N}$.

   Solution Taking $m = n = 1$, we get $f(1 \cdot 1) = f(1)f(1)$

and hence $f(1) = 1$. Also, $f(2n) = f(2)f(n)$ and $2n + 1$ divides $f(2n) + 1 = f(2)f(n) + 1$. This gives that $\gcd(f(2), 2n + 1) = 1$, for all n. This means that $f(2) = 2^k$ for some natural number k. Since $3 = 1 + 2$ divides $f(1) + f(2) = 1 + 2^k$, it follows that k must be odd.

Now take an arbitrary power 2^m and an arbitrary natural number n. Since $2^m + n$ divided $f(2^m) + f(n) = f(2)^m + f(n) = 2^{mk} + f(n)$, it follows that $2^m + n$ divides $2^{mk} + f(n)$. But

$$\begin{aligned} 2^{mk} + f(n) &= (2^{mk} + n^k) + (f(n) - n^k) \\ &= M(2^m + n) + (f(n) - n^k) \end{aligned}$$

where M is an integer ($2^{mk} + n^k$ is divisible by $2^m + n$ since k is odd). This gives that $2^m + n$ divides $f(n) - n^k$ for all m and hence $f(n) - n^k = 0$. Thus $f(n) = n^k$.

ASSOCIATION ACTIVITIES

General Body meeting

Minutes of the general body meeting held or 28.12.2017 at the venue of 52nd conference of the AMTI

37 members were present.

1. After prayer by Ms. Lakshmi, the President Prof. I.K Rana welcomed the gathering after the General Secretary requested him to preside over the meeting.

2. Minutes of the last General Body meeting was then read by the General Secretary. It was adopted unanimously.

3. Audited A/C for the year 2016-2017 was then perused by the members and it was also adopted unanimously after few doubts raised and answered by the general secretary on behalf of the Treasurer. The suggestion to keep fixed fund for publication and scholarship was agreed to be processed in consultation with the auditor.

4. In the publication of MT , JM by web it was discussed to explore possibilities of doing so after ensuring PDF disseminated through Whatsapp as suggested by Sri Ramesh, Sri Bharath Karmakar and Sri Sivaraman.

5. In that connection an appeal was made to the members to send articles for publication in MT / JM on experiments, action research etc., as was seen in ELTAI journal.

6. Then Election to the Executive Committee for the next term took place.

a. Prof. I. K. Rana was reelected as President unanimously.

b. The following members were elected for the Executive Committee apart from existing members with the byelaws permitting up to 25 and others, if found necessary as special invitees.

i) Ms. Jayapriya, Jain Heritage School, Bangalore, proposed by Ms. Archana and accepted by others.

ii) Sri Ramesh, Jain Heritage School, proposed by Dr. Palanivasan and accepted by others.

iii) Dr. R.Sivaraman, proposed by Sri Mahadevan and accepted by others.

iv) Sri Kumarasamy, proposed by Prof by Dr. M. Palanivasan and Sri R. Athmaraman and accepted unanimously by others.

To the suggestion of Sri Bharat Karmarkar on role of Executive Committee member, particularly outside head quarters it was opined by the president that they were expected to expand AMTI influence in their place by understanding and co-operation as added by the General Secretary.

7. Sri Govinda Reddy and Ms. Archana wanted at least 3 to 4 workshops to be conducted in their places. They were informed to send proposals to the AMTI office to identify resource persons and plan the workshops.

8. a) The absence Sri V. Seshan, treasurer, due to KVS assignment at Hyderabad was noted.

b) To the proposal of Sri Bharath Karmarkar on workshop in project office in May to initiate project oriented learning for teachers, it was agreed in principle to be decided by the Executive Committee after fixing dates without clash of our regular programme with suitable planning including finance.

9. On Dr. S. Muralidharan's suggestion to start web based orientation for Olympiad aspirants anywhere it was agreed in principle by everyone. The Executive Committee may decide the details of the offer and modalities as required to be formalized and announced through our website.

10. The following persons have volunteered to join this project. a. Dr. S. Muralidharan b. Ms. Jayapriya c. Sri Ramesh d. Ms. Radha Balakrishna e. Ms. Uma Iyer f. Sri Bharath Karmarkar g. Ms. R Shanthi h. Ms. Pushpalatha

11. The General Secretary placed his proposal to retire from the post with effect from 01.04.2018, at the end of the present term after more than 20 years of the work. There were discussions on how and who the successor should be decided. Then it was agreed by the majority approved by general body members present and as agreed by most of elected Executive Committee members present to accept the proposal to give the office of General Secretary to Ms. R. Shanthi. She accepted and assured her best services. Sri M. Mahadevan thanked all present for the understanding cooperation extended to him during his term. Sri S.R Santhanam, Sri Kumarasamy suggested that he should function as advisor and he agreed.

12. With vote of thanks by the general secretary and National Anthem, the meeting came to an end.

Executive Council meeting

Minutes of the Executive Committee meeting held at Sri Sarada Vidyalaya on 10-02-2018

1. Due to the delayed arrival of the Executive Chairman, Sri V.Seshan was requested to preside over the meeting. He did so and welcomed the members and directed the General Secretary to take up the agenda items one after the other. The following members attended the meeting:

Dr. R. Santhi, Smt. C. Jayashree, Sri G. Gnanasundaram, Sri. B.N. Shivashankar, Sri V. Sundaramurthy, Dr. P. Bakthavatchalu, Dr. S. Muralidharan, Prof. G.P. Youvarja, Sri P.V. Swaminathan, Dr. S.R. Santhanam, Sri R. Athmaraman, Sri V. Seshan, Dr. M. Palanivasan, Sri P. Ramesh, Prof. J. Pandurangan, Dr. M.Kumarasamy, Sri M. Mahadevan.

The General Secretary then took up the points one after the other:

i) The journal The Mathematics Teacher had already seen the two issues released for the year 2017-18, Vol. 53 (1&2) and Vol.53 (3&4). He displayed the latest released one.

ii) The journal Junior Mathematician saw two of the three issues already released with the third awaited in April 2018.

iii) The draft of call notice for NMTC was ready as approved by the Talents Secretary with the note on individuals to identify people for screening test.

iv) On conference, Sri S.R.Santhanam agreed to visit Kota to see the claim of the Disha Delphi School and decision was deferred pending the same. Besides Rajamundrys claim also was noted.

v) On May workshops the draft call notice as approved by the Project secretary was noted as published in the web. It was also noted that already about 20 registrations have reached the office.

vi) On popular lectures the EC noted the process to improve the identification of venues and speakers and expected the new in-charge to do the needful.

2. The financial position as on 31st January was noted by all members. The proposals of the General Secretary to equip the project office with a projector for the second class room, related darkening of the room(s) and Sri V.Seshans suggestion of providing audio aids to reach all the students as also inverter provision for lights and fans for managing the power disruption during the workshops were approved. The suggestion to get generator being costly and the suggestion for taking it on rent were deferred till details are recd.

3. The request of Dr. Bhaktavatsalu to continue in the EC was accepted.

4. The minutes of the GB that took place in Bengaluru was noted to be presented by the new General Secretary, during the next conference.

5. Then allocation of duties to various members took place.

i) President has already been elected in the General Body - Prof. I.K. Rana, continuing.

ii) The executive Chairman opined to continue if the EC feels so and it was unanimously accepted.

iii) General Secretary had also been elected at the General Body Dr. K. Santhi and noted by the EC.

iv) On treasurer Sri V. Seshan who wanted to be relieved by 31st January was requested continue till 31st March. There were two candidates competing - one Sri V. Sundaramurthy proposed by Dr. S. Muralidharan and Sri S.R. Santhanam and Dr. Palanivasan self volunteered and endorsed by Dr. Bhakatavatsulu. On the suggestion that if one volunteers to withdraw it will be good but none did so. There was an election with 6 votes to Sri V. Sundaramurthy and 4 to Dr. M. Palanivasan. Sri V. Sundaramurthy was declared elected as treasurer.

v) The existing officials - Secretary for Talents exam, Editor for MT and JM, agreed to continue.

vi) With Dr. Palanivasan not agreeing to continue as project secretary Dr. Bhaktavatsulu was offered and he agreed.

vii) Sri G. Gnanasundaram was requested to associate with Sri Athmaraman in the preparation of JM so that later he can take over. He agreed to do so.

viii) Sri Ramesh was elected to be secretary for popular lectures and he agreed to do so.

ix) Dr. Youvaraj of Ramanujan Institute, Dr. Swaminathan were taken into the executive committee.

6. The request of Smt. Aruna, office assistant, to grant an advance of Rs. 5 lakhs for her land acquisition there was general discussion and finally it was agreed to

i) Grant her Rupees three lakhs as advance to receive it back in installments of Rs. 15,000/- each month debited from her salary with bank interest added, making her insurance as collateral security.

ii) This is manageable from our reserves.

7. The Retiring General Secretary thanked all the members particularly the treasurer, Sri V. Seshan, for the abundant confidence they have reposed in him which helped him to report better turn over all these years enabling him to retire with satisfaction.

8. Sri R. Athmraman narrated his association with the AMTI almost since its inception and added the contribution of the General Secretary especially for the improved financial position of the AMTI.

9. Sri M. Mahadevan, the outgoing secretary and Dr. R. Shanti thanked all the members for the success of the meeting.

Farewell to Mr M Mahadevan
General Secretary, AMTI

A farewell function to Sri M. Mahadevan, who had been the General secretary, AMTI, for the longest tenure of more than two decades, was organized on 31st March 2018 at our Project Office, Alappakkam. Apart from the members from various states, many founder members and office bearers of AMTI participated in this event to honor Sri M. Mahadevan. A citation was presented to Sri Mahadevan for his selfless and exemplary service as the General Secretary. The invitees shared their personal and professional experiences and memorable moments with Sri Mahadevan, highlighting his academic, administrative and leadership skills. It was an emotional moment to every one when Mr. Mahadevan addressed the gathering and also expressed his plan to donate a small portion of his personal property to AMTI. The event was concluded with the vote of thanks by the incoming secretary Dr. R. Shanthi, followed by lunch to all the participants.
The citation presented to Mr Mahadevan is given below.

M MAHADEVAN

Mr Mahadevan has led AMTI with distinction as its general Secretary, for the past 21 years. His retirement today will mark the end of an era of making great strides in the development and popularity of AMTI.

Mr Mahadevan can be aptly described as Leader extraordinaire. He came in contact with AMTI during 1967 in a mathematical expo organized by Mr P K Srinivasan. He was actively involved in the activities of AMTI during the past 50 years. He became an executive committee member in 1977 and the treasurer in 1991. He was elected unanimously as General Secretary in 1997 and it was a great turning point in the history of AMTI.

When he took over as Secretary, the association had a one room office, the revenue resources were meagre, not many volunteers to share the work and not to mention, the mathematics competition culture was all but absent among the students. With his extraordinary abilities, his magic wand changed all that. He dreamed big for AMTI and ensured that all his dreams come true. First and foremost, he enhanced the revenue source of AMTI the talent competition participation from a mere 8000 students to 1,40,000 students today. While attempting to build a bigger office for AMTI, on the land kindly donated by Mr Santhanam of SSM Group of institutions, the initial donation of Rs 50000 was from Mr Mahadevans brother. The second office was built entirely from the competition revenue. The building currently having two floors with dormitories for students, guest rooms for faculty, air-conditioned class rooms, a computer lab with 20 computers, a well-stocked library with more than 5000 books and a reading room at a cost of nearly 1 crore - stand testimony to his judicious use of funds to leap frog AMTI to an organization of reputation.

The talent tests for grades primary, sub junior and senior levels, in addition to the inter and junior levels were introduced during his period. He initiated the concept of Math workshops for teachers and students. These have become popular and schools from many other states (Arunachal Pradesh, Orissa, Gujarat, Andhra Pradesh etc) seek AMTI services for conducting workshops for their students. The Annual conference of the association is attended by more than 300 participants and we receive offers from various schools to host our conference in their place.

His enthusiasm while talking about AMTI is something seen to be believed. He describes the activities of AMTI with as much passion as a mother describing her

childs great qualities. He is an excellent teacher and inspires students with his down-to-earth techniques for communicating.

We thank you, Sir, for bringing AMTI to the dizzying heights. It will be very difficult for anyone to show as much dedication, constancy of purpose, tenacity in execution as you have demonstrated. But rest assured, we will collectively emulate you and try our best to take AMTI still further. That would be the only way we can express our gratitude to what you have accomplished for AMTI.

A teacher never retires. Please continue to guide us to do good things and chide us when we do something not so good.

We wish you a very happy and peaceful retirement. May God bless you with good health and peace. You will remain in our hearts forever and we all cherish the experience we had with you.

Some Moments During Farewell To Shri Mahadevan

THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

(Founded 1965; Registered as S.No.143 of 1965 under the Societies Registration Act XXI of 1860)

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To